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Construction of Instantons and Skyrmions in dimensions higher than four

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Abatract

Instantons are solutions to a Yang-Mills model. The four-dimensional, one time and three spaces, Yang-Mills model is a gauge theory which describes the behavior of the fundamental interactions without the gravitation, namely the electromagnetic, weak and strong interactions. It is know that instantons play important roles in the study of non-perturbative effects in gauge theories. Of particular importance for the instantons is its systematic generation method of solutions, known as the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction. We usually need the solve partial differential equations (PDEs) to obtain the instantons, but when using the ADHM construction then we suffice to only solve algebraic equations instead of PDEs. Moreover it is know that the ADHM construction can algebraically constructs all the instantons.

One of other important solution in particle physics is known as a Skyrmion. The Skyrmions are solutions to a four-dimensional (static) Skyrme model which is a model for element particles in the low-energy effective theory of the strong interaction. However, no analytic solutions of Skyrmion have been found yet, the numerical solutions are only known. Finding proper solutions of Skyrmions is a long-standing problem. There are several directions to construct solutions. For example, the rational map ansatz provides a good approximation to the Skyrmion solutions. Alternatively, there is another promising approach to Skyrmions known as an Atiyah-Manton construction. The Aityah-Manton construction gives well approximated static Skyrmion solutions from the holomomy of the Yang-Mills instantons.

We sometimes consider extra dimensional models, for example the Kaluza-Klein theory, the brane world scenario, the M-theory and others, to solve some modern physics problems. Hence, it is natural that we consider instantons and Skyrmions in higher dimensions. Indeed, several kinds of higher-dimensional “instantons” were proposed, and these have been studied in various contexts. Similary it is interesting topic that we study generalization of construction methods of instantons and Skyrmions in higher dimensions. This paper treats mainly the higher-dimensional ADHM construction of self-dual type instantons and the Atiyah-Manton construction of higher-dimensional Skyrmions.

Contents

Introduction

This paper discuss about instantons and Skyrmions in higher dimensions. Particularly we attempt the systematic construction of the instantons and the Skyrmions in dimensions higher than four.

The original instantons are defined as topological solitons of pure Yang-Mills theory defined in four-dimensional Euclidean space. Yang-Mills theory is a gauge theory based on the compact, reductive Lie group, and is the basis of our understanding of the Standard Model of particle pysics. This idea can date back to the attemption of unified with the general relativity and electromagnetism by H. Weyl in 1918 [1]. The original Weyl idea is that "the physical laws is invariant under the change of scale (or "gauge") and have also a local symmetry of general relativity" (This is the original of the name "gauge theory"). Unfortunately this original idea did not work well, but recently gauge theory can be regarded as the expansion of this idea to the internal space. In 1954, C. N. Yang and R. Mills first introduced a non-abelian gauge theory as a model of the SU(2) symmetry group on the isospin doublet of protons and neutrons [2]. As the same time, W. Pauli, A salam and R. Utiyama independently introduced a non-abelian gauge theory respectively. Particularly the theory that was introduced by R. Utiyama was generalized gauge theory including the other non-abelian gauge groups [3], and his theory also descript general relativity by the frame of gauge theory. In 1961 S. L. Glashow discovered a way to unifine with the electromagnetic ans weak interactions [4]. In 1967 S. Weinberg [5] and A. Salam [6] incorporated the Higgs mechanism into Glashow's model, giving it its modern form. This electroweak unified theory is descripted by a $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge theory, is called as Glashow-Weinberg-Salam theory. The strong interaction is described by a SU(3) gauge theory, and the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge theory that direct combine this SU(3) gauge theory and the electroweak unified theory is know as the standard model of particle physics. These three forces, electromagnetic, weak and strong interactions, can be described by the gauge theories, thus we naturally consider that these forces can be merged into one single force, namely a one larger gauge group. This attemption is called as a Grand Unifined Theory (GUT), and various GUT models are proposed.

The Yang-Mills theory describe the interactions, hence the solutions to the Yang-Mills theory are important to understand the behavior of the gauge bosons, namely the interactions. Instantons are one of the classical solutions to the Yang-Mills theory, and are the solutions of the minumum action in the four-dimensional Euclidean space. The physical motivation for considering four-dimensional Euclidean space is that in quantum field theory in (3+1)-dimensional Minkowski space-time one is led to the computation of path integrals which need to be analytically continued in order to be well defined. The reason classical solutions are important is that they dominate the path integral, and in particular the instanton solutions generate non-perturbative quantum effects. The instantons are also topological solitons which are stable, particle-like objects, with finite energy and a smooth structure. A salient feature of the instantons is that there is systematic construction of solutions, known as the ADHM construction which is introduced by Atiyah, Drinfeld, Hitchin and Manin [7]. We usually need the solve partial differential equations (PDEs) to obtain the instantons, but when using the ADHM construction then we suffice to only solve algebraic equations instead of PDEs. The ADHM construction also reveals the Kähler quotient structure of the instanton moduli space and provides the scheme to calculate the non-perturbative corrections in the path integral.

One of other topological solitons is known as Skyrmions. The Skyrmions are solutions to an (static) Skyrme model which was proposed by T. Skyrme in 1962 [8]. Skyrme believed that the nucleons (protons and nuetrons) in a nucleus were moving in a nonlinear, classical pion medium. This made him reconsider the pion interaction terms. Symmetry arguments led to a particular form of Lagrangian for the three-component pion field, coresponding to the spinless pions (π^+, π^-, π^0), with a topological structure which allowed a topologically stable soliton solution of the classical field equation, distinct from the vacuum. This model is the Skyrme model, and this Lagrangian is essentially a one-parameter model of the nucleon. Fixing the parameter with proton radius, and also gives all other low-energy properties, which appear to be correct to about 30% (which compared with experiment values) [9]. It is this predictive power of the model that makes it so appealing as a model of the nucleon. Although it is hard to generally solve the equation of motion of the Skyrme model, there are some methods to lead a good approximation of the Skyrmions. One of this methods is an approach that using the holonomy of the instantons, known as an Atiyah-Manton construction [10]. The Skyrmions from the instantons hold some moduli pamameters, thus it is advantageous for study the behaviour of the Skyrmions.

We sometimes consider extra dimensional models, for example the Kaluza-Klein theory, the brane world scenario, the M-theory and others, to solve some modern physics problems. Hence it is natural that we consider the topological soliton in higher dimensions. Indeed, several kinds of higher-dimensional “instantons” were proposed, and these have been studied in various contexts. Similary it is interesting topic that we study generalization of construction methods of topological solitons in higher dimensions. This paper treats mainly the higher-dimensional ADHM construction of self-dual type instantons and the Atiyah-Manton construction of higher-dimensional Skyrmions.

This paper is organized the two parts, the first part is about the ADHM construction and instantons, the second part is about the Atiyah-Manton construction and Skyrmions. These parts contain the two chapters respectively, the firsts are reviews in usual four-dimensions, the seconds are discussion in higher dimensions. Contents of the each chapter are summarized at the beginning of each chapter.

Part I

The ADHM constructions and instantons

Chapter 1

In four dimensions

In this chapter, we give a review on the ADHM construction and the instantons in the usual four dimensions.

The instantons in four dimensions are defined as solutions to the (anti-)self-dual(ASD) equation $F = \pm *_4 F$. Here F is the field strength 2-form of the gauge field with a gauge group G and the symbol $*_d$ is the Hodge dual operator in d-dimensional Euclidean space. It is well known that instantons play important roles in the study of non-perturbative effects in gauge theories [11, 12]. Through the Bianchi identity, instantons are solutions to the equation of motion in the four-dimensional pure Yang-Mills theory. The instantons are, moreover, characterized by the homotopy group $\pi_3(G)$, therefore we can classify these by the second Chern number $c_2 = \frac{1}{8\pi^2}\int \mathrm{Tr}[F \wedge F]$. Of particular importance for the instantons is its systematic generation method of solutions, known as the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [7]. The ADHM construction algebraically constructs whole the instantons in four dimensions, and the quaternion plays central roles on this algebraic side.

It is well known that the instantons are related to other lower dimensional solitons. For instance, a caloron is the soliton solution that we take a periodic direction in the instanton [13]. The dimensional reduction of the ASD equation to three dimensions leads to the Bogomol'nyi equation, and the Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles are defined as the solutions to this equation [14]. In this paper, monopoles mean the BPS monopoles. There is the systematic construction, which is similar to the ADHM construction, of the monopoles and the calorons. This construction is called the Nahm construction [15]. In three dimensions, there is an another soliton, known as Skyrmion, which is the solution of the static Skyrme model. The Atiyah-Manton construction produces well-approximated solutions of the Skyrmions from the instantons [10]. An higher-dimensinal caloron and an Atiyah-Manton construction are discussed in the below chapters.

The organization of this chapter in as follows. Section 1.1 is introduction of the Yang-Mills model and SU(2) instantons. Section 1.2 is reviewed the ADHM construction of the instantons with U(2) gauge group.

1.1 Yang-Mills model and instanton in four dimensions

The action of the (pure-)Yang-Mills model is given by

$$S = -\frac{1}{2}\int \mathrm{Tr}\left[F \wedge *_4 F\right], \tag{1.1}$$

where $F := \frac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu$ is the field strength 2-form of the gauge field with a gauge group G and $*_4$ is the Hodge dual operator in four dimensions. $F_{\mu\nu}$ is defined by $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and A_μ is the anti-Hermite gauge field ($A_\mu^\dagger = -A_\mu$) which takes value in a Lie algebra $\mathcal{G}$. If we choose the Hermite gauge field ($A_\mu^\dagger = A_\mu$) then the definition of field strength is replaced to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and the action coefficient signature is replaced 1 instead of -1. The Lie algebra $\mathcal{G}$ is associated with the non-Abelian gauge group G and the greek indices $\mu, \nu, \cdots = 1, 2, 3, 4$ are the four-dimensional time-space indices. The Euler-Lagrange equation of (1.1) called as a Yang-Mills equation is

$$D_\mu F^{\mu\nu} := \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{1.2}$$

In the case of Hermite gauge, we have to replace the anti-commutative term $[A_\mu, F^{\mu\nu}]$ to $-i[A_\mu, F^{\mu\nu}]$. The Yang-Mills equation is a nonlinear partial differential equation, thus it is a difficulty which we solve to the Yang-Mills equation analytically in general. In more detail, when the basis manifold is Minkowski space-times then the Yang-Mills equation becomes the hyperbolic type partial differential equation, and it is known that in mathematical sense the hyperbolic type differential equation is very difficult to solve. On the other hand, when the basis manifold is Euclidean space the Yang-Mills equation becomes the elliptic type differential equation, and the elliptc type equation is slightly easy than the hyperbolic type equation.

Instanton is one of the solutions to the Yang-Mills equation (1.2) in the four-dimensional Euclidean space $\mathbb{R}^4$. We note that we discuss the only Euclidean spaces in the following, so we ignore the position of space indices (i.e. upper indices are same means as down indices). In Euclidean space, the Yang-Mills equation is elliptic type equation. Still, it is difficult to solve the Yang-Mills equation because it is the second order partial differential equation. On the other hand, an ASD equation is first order differential equation, thus we often solve it. The ASD equation is given by

$$F_{\mu\nu} = \pm *_4 F_{\mu\nu}, \tag{1.3}$$

where $*_4 F_{\mu\nu}$ is the Hodge dual of the field strength which is defined by $*_4 F_{\mu\nu} = \frac{1}{2!}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ ($\Longleftrightarrow$ $*_4 F = *_4 F_{\mu\nu} dx^\mu \wedge dx^\nu$). We sometimes call this equation that is positive signature as the self-dual equation. On the other hand, this equation that is negative signature as the anti-self-dual equation. Instantons are defined as the solutions to the ASD equation (1.3), and are solutions to the Yang-Mills equation (1.2) through the bianchi identity:

$$D_\mu *_4 F_{\mu\nu} = 0, \tag{1.4}$$

This equation is an identity equation, thus the solutions to the ASD equation (1.3) automatically satisfies the Yang-Mills equation (1.2).

In particular importance, the instantons give the minimum action of the Yang-Mills model (1.1). The Bogomol'nyi completion of the Yang-Mills action is

$$S = -\frac{1}{4}\int \mathrm{Tr}\left[(F \mp *_4 F)^2 \pm 2F \wedge F\right] \geq \mp\frac{1}{2}\int \mathrm{Tr}\left[F \wedge F\right], \tag{1.5}$$

where we have defined

$$(F \mp *_4 F)^2 = (F \mp *_4 F) \wedge *_4 (F \mp *_4 F)\,. \tag{1.6}$$

The Bogomol'nyi bound of the action (1.1) is saturated when the solutions satisfy the ASD equation (1.3). Then the action is bounded from below by $S = \mp\frac{1}{2}\int \mathrm{Tr} F \wedge F = \pm 4\pi^2 Q$. Here Q is the second Chern number:

$$Q = -\frac{1}{8\pi^2}\int \mathrm{Tr} F \wedge F. \tag{1.7}$$

Therefore the instantons are classified by the second Chern number and the Chern number is sometimes called as instanton (topological) charge or instanton number. It is particularly importance that the homotopy of the gauge group $\pi_3(G)$ is associated with the number of the instanton variety. For example, when the rank of the (special) unitary gauge group N is one then the homotopy group became trivial: $\pi_3(\mathrm{U}(1)) = 0$, hence the instanton (that is topological soliton) of U(1) gauge group does not exist. We are intersted in instantons that are characterized by the instanton number, namely instanton becomes the topoligical soliton, thus we consider the non-trivial homotopy group. The some examples of the non-trivial homotopy groups for classical groups are

$$\pi_3\left(\mathrm{U}(N)\right) = \pi_3\left(\mathrm{SU}(N)\right) = \mathbb{Z}, \quad N \geq 2, \tag{1.8a}$$

$$\pi_3\left(\mathrm{SO}(4)\right) = \mathbb{Z} \oplus \mathbb{Z}. \tag{1.8b}$$

In the case of $\pi_3(G) \neq 0$, the second Chern number is integer and the instantons with the gauge G are completely classified by this integer number. The solution to self-dual equation is called as instanton and this instanton charge is positive. On the other hand, the solution to anti-self-dual equation is called as anti-instanton and this instanton charge is negative. The instanton has the property of the self-duality to the gauge field strength F. Next we consider this property on the differential form.

In four-dimensional Euclidean space (which metric is $ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu$), the action of the Hodge dual operator with 2-from is

$$*_4 \left(dx^\mu \wedge dx^\nu\right) = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} dx^\rho \wedge dx^\sigma. \tag{1.9}$$

Therefore the operator that the Hodge dual operator acting twice is the identity operator [1]: $*_4^2 = 1$. Hence the Hodge dual operator is the automophism of the 2-form linear space $\Lambda^2(T_x^*\mathbb{R}^4)$. Now $T_x\mathbb{R}^4$ means the tangent space on $x \in \mathbb{R}$ and $T_x^*\mathbb{R}^4$ means

[1] Note that the Hodge dual operator is dependent on the space metric. For example, the twice Hodge dual: $*^2$ in the four-dimensional Minkowski space (sig$(- + ++)$) becomes $*^2 = -1$, and in the four-dimensional hyperbolic space (sig$(- - ++)$) becomes $*^2 = 1$.

the dual tangent space on x. For this reason and $*_4^2 = 1$, the 2-form linear space $\Lambda^2(T_x^*\mathbb{R})$ is able to be direct sum denompose with the eigenspace that the eigenvalue is ± 1:

$$\Lambda^2(T_x^*\mathbb{R}^4) = \Lambda_+^2(T_x^*\mathbb{R}^4) \oplus \Lambda_-^2(T_x^*\mathbb{R}^4). \tag{1.10}$$

where $\Lambda_\pm^2(T_x^*\mathbb{R}^4) := \left\{\omega \in \Lambda^2(T_x^*\mathbb{R}^4) \mid *_4\, \omega = \pm\omega\right\}$. Therefore the ASD equation means that the self-dual or anti-self-dual part of the field strength $F \in \Lambda^2(T_x^*\mathbb{R}^4)$ is vanished. In mathematically, this fact means that the gauge field can be written as the framework of twistor theory. This framework was first introduced by Ward [16], who demonstrated that the twistor transform of Penrose could be used to provide a correspondence between instantons and certain holomorphic vector bundles over the twistor space. There are two alternative methods for constructing the appropriate bundles; the first involves obtaining the bundle as an extension of line bundles and leads to the Atiyah-Ward construction [17], whereas the second is the method with using monads known as the ADHM construction [7].

1.1.1 SU(2) instantons

In this subsection, we review some instantons with the SU(2) gauge group.

First of all, we introduce the 't Hooft symbol which play central role in SU(2) instantons:

$$\eta_{\mu\nu}^{(+)} = e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu, \quad \eta_{\mu\nu}^{(-)} = e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger, \tag{1.11}$$

where $e_i, e_i^\dagger$ are defined by

$$e_\mu = \delta_{\mu 4}\mathbf{1}_2 - i\delta_{\mu i}\sigma_i, \quad e_\mu^\dagger = \delta_{\mu 4}\mathbf{1}_2 + i\delta_{\mu i}\sigma_i. \tag{1.12}$$

Here $i = 1, 2, 3$ and σ_i are the Pauli matrices. The most important property of the 't Hooft symbol is that it satisfies the ASD relation:

$$\eta_{\mu\nu}^{(\pm)} = \pm\frac{1}{2!}\varepsilon_{\mu\nu\rho\sigma}\eta_{\rho\sigma}^{(\pm)}. \tag{1.13}$$

The parameters that are included in the instantons are called the muduli parameters, and the number of these parameters depend the instanton number and the gauge group. The dimension of the instanton moduli space, namely max moduli number, is calculated by using the result of the Atiyah-Singer index theorem [18, 19]. In SU(2) (or U(2)) gauge group, the full instanton moduli number is $8k - 3$.

BPST instanton

The Belavin-Polyakov-Schwartz-Tyupkin(BPST) instanton [11] is most simplify instanton. The gauge field of the BPST instanton is given by

$$A_\mu = -\frac{1}{2}\frac{\tilde{x}^\nu}{\lambda^2 + ||\tilde{x}||^2}\eta_{\mu\nu}^{(\pm)}, \tag{1.14}$$

where we have defined $\tilde{x}^\mu := x^\mu - a^\mu,\ a^\mu \in \mathbb{R}$ is the position of the instanton, $\lambda \in \mathbb{R}$ is the instanton size and $||\tilde{x}||^2 = (x^\mu - a^\mu)(x_\mu - a_\mu)$. This field strength is evaluated to

$$F_{\mu\nu} = \frac{\lambda^2}{(\lambda^2 + ||\tilde{x}||^2)^2}\eta_{\mu\nu}^{(\pm)}. \tag{1.15}$$

For (1.13), the BPST instanton's field strength is manifestly satisfies the ASD equation (1.3).

The topological charge, namely the second Chern number (1.7) is rewritten as

$$Q = -\frac{1}{8\pi^2}\int \mathrm{Tr}F \wedge F = -\frac{1}{8\pi^2}\int d^4x\mathrm{Tr}\left[\left(\frac{1}{2}\right)^2 \varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right], \tag{1.16}$$

thus the topological charge of the BPST instanton becomes

$$Q = -\frac{1}{8\pi^2}\frac{1}{4}\int d^4x\left(\frac{\lambda^2}{(\lambda^2 + ||\tilde{x}||^2)^2}\right)^2 \mathrm{Tr}\left[\varepsilon_{\mu\nu\rho\sigma}\eta_{\mu\nu}^{(\pm)}\eta_{\rho\sigma}^{(\pm)}\right] = \pm\frac{1}{8\pi^2}\frac{1}{4}\frac{\pi^2}{6}4 \cdot 4! \cdot 2 = \pm 1, \tag{1.17}$$

where we use $\varepsilon_{\mu\nu\rho\sigma}\eta_{\mu\nu}^{(\pm)}\eta_{\rho\sigma}^{(\pm)} = \mp 4 \cdot 4!\mathbf{1}_2$.

The BPST instanton has four space coordinate moduli parameters a^μ and one size moduli parameter λ, thus the moduli parameters of the BPST instanton total to five. This number is same as 1-instanton full moduli parameter $(8 - 3 = 5)$, hence 1-instanton is whole representation by the BPST instanton.

’t Hooft and JNR k-instanton

When we consider a multi-instantons we usually use the following ansatz:

$$A_\mu = \frac{1}{4}\eta^{(\mp)}_{\mu\nu}\partial_\nu \ln\rho(x). \tag{1.18}$$

This ansatz is callled as the Corrigan-Fairlie-’t Hooft-Wilczek(CFtHW) ansatz. Substituting this ansatz into the ASD equation lead to the four-dimensional Laplace equation:

$$\partial_\mu\partial_\mu\rho(x) = 0. \tag{1.19}$$

There are two types of the well-known multi-instanton solutions to this equation, known as ’t Hooft instantons [12] and Jackiw-Nohl-Rebbi(JNR) instantons [20].

The ’t Hooft k-instanton is

$$\rho(x) = 1 + \sum_{m=1}^{k}\frac{\lambda_m^2}{\|\tilde{x}_m\|^2}, \tag{1.20}$$

where $\tilde{x}^\mu_m := x^\mu - a^\mu_n$ and $\|\tilde{x}_m\|^2 = \tilde{x}^\mu_m\tilde{x}^\mu_m$. The moduli parameters of the ’t Hooft instantons have the simple physical meaning: $\lambda_m \in \mathbb{R}$ denotes each instanton size and $a^\mu_m \in \mathbb{R}$ denotes each instanton position. Note that the ’t Hooft one-instanton is singular at the instanton position:

$$A^{\text{singular}}_\mu = \frac{1}{4}\eta^{(\mp)}_{\mu\nu}\partial_\nu \ln\left(1 + \frac{\lambda^2}{\|\tilde{x}\|^2}\right), \tag{1.21}$$

while the BPST instanton (1.14) discussed in the previous subsubsection is non-singular. These solution are connected by the following singular gauge transformation:

$$A^{\text{non-singular}}_\mu = g_1 A^{\text{singular}}_\mu g_1^{-1} + g_1\partial_\mu g_1^{-1}, \quad g_1 = \frac{-\tilde{x}}{\sqrt{\|\tilde{x}\|^2}}, \tag{1.22}$$

where $\tilde{x} := (x^\mu - a^\mu)e_\mu$.

The JNR k-instanton which is another solution to (1.19) is

$$\rho(x) = \sum_{m=0}^{k}\frac{\lambda_m^2}{\|\tilde{x}_m\|^2}. \tag{1.23}$$

The moduli parameters of the JNR instantons do not have the simple physical meaning.

1.2 The ADHM construction in four dimensions

In this section, we give a brief review on the ADHM construction of instantons in four dimensions. We consider of the gauge group U(2).

The four-dimensional Weyl operator $\Delta_{(4)}$ is defined by

$$\Delta_{(4)} = C(x\otimes \mathbf{1}_k) + D, \tag{1.24}$$

where C and D are quaternionic $(k+1)\times k$ matrices, k is the instanton charge and $x = x^\mu e_\mu$. Here x^μ $(\mu = 1,2,3,4)$ is the Cartesian coordinate of the four-dimensional Euclid space, $e_\mu = (-i\sigma_i, \mathbf{1}_2)$ is the basis of the quaternion and σ_i are the Pauli matrices. The Weyl operator $\Delta_{(4)}$ is assumed to satisfy the ADHM constraint:

$$\Delta^\dagger_{(4)}\Delta_{(4)} = \mathbf{1}_2\otimes E_k, \tag{1.25}$$

where $\Delta^\dagger_{(4)}$ is the quaternionic conjugate of $\Delta_{(4)}$ and E_k is an invertible $k\times k$ matrix.

In order to construct the instanton solution for the gauge field $A_\mu(x)$, it is necessary to find a quaternionic $(k+1)$ column vector $V(x)$ obeying the Weyl equation:

$$\Delta^\dagger_{(4)}V(x) = 0, \tag{1.26}$$

where $V(x)$ is the zero-mode normalized as $V^\dagger(x)V(x) = \mathbf{1}_2$. The gauge field $A_\mu(x)$ of instantons is given by

$$A_\mu(x) = V^\dagger(x)\partial_\mu V(x). \tag{1.27}$$

Using the expression (1.27), the field strength is calculated such as

$$F_{\mu\nu} = \partial_\mu V^\dagger \left(\mathbf{1}_{2+2k} - VV^\dagger\right)\partial_\nu V - (\mu \leftrightarrow \nu). \tag{1.28}$$

Now use the completeness relation:

$$\mathbf{1}_{2+2k} - VV^\dagger = \Delta_{(4)}(\Delta^\dagger_{(4)}\Delta_{(4)})^{-1}\Delta^\dagger_{(4)}, \tag{1.29}$$

then (1.28) is rewritten as

$$\begin{aligned} F_{\mu\nu} &= V^\dagger C(e_\mu \otimes \mathbf{1}_k)(\Delta^\dagger_{(4)}\Delta_{(4)})^{-1}(e^\dagger_\nu \otimes \mathbf{1}_k)C^\dagger V - (\mu \leftrightarrow \nu) \\ &= V^\dagger C(\Delta^\dagger_{(4)}\Delta_{(4)})^{-1}\left(\eta^{(-)}_{\mu\nu} \otimes \mathbf{1}_k\right)C^\dagger V, \end{aligned} \tag{1.30}$$

where we have used the ADHM constraint (1.25). Now we recall that the 't Hooft symbol $\eta^{(-)}_{\mu\nu}$ satisfies the anti-self-dual relation (1.13). Therefore the field strength $F_{\mu\nu}$ associated with the solution (1.27) automatically satisfies the ASD equation $F = \pm *_4 F$.

For the above discussion, we find that a key point of the ADHM construction is that the 't Hooft symbol $\eta^{(\pm)}_{\mu\nu}$ which is constructed from the basis e_μ satisfies the ASD relation. Therefore if we formulate the ADHM construction of instantons in higher dimensions then we need to find the basis that satisfies the ASD relation in higher dimensions.

Let us analyze the ADHM constraint in more detatil. For (1.24), the ADHM constraint (1.25) becomes

$$(x^\dagger \otimes \mathbf{1}_k)C^\dagger C(x \otimes \mathbf{1}_k) + (x^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(x \otimes \mathbf{1}_k) + D^\dagger D = \mathbf{1}_2 \otimes E_k(x). \tag{1.31}$$

Since the ADHM constraint hold for all $x \in \mathbb{R}^4$, this constraint must be satisfied by the each order term of x.

$$^\forall x \in \mathbb{R}^4,\ \ \Delta^\dagger_{(4)}(x)\Delta_{(4)}(x) = \mathbf{1}_2 \otimes E_k(x) \iff {}^\forall x \in \mathbb{R}^4,\ \ \begin{cases} (x^\dagger \otimes \mathbf{1}_k)C^\dagger C(x \otimes \mathbf{1}_k) & = \mathbf{1}_2 \otimes E^{(1)}_k(x^2), \\ (x^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(x \otimes \mathbf{1}_k) & = \mathbf{1}_2 \otimes 2E^{(2)}_k(x), \\ D^\dagger D & = \mathbf{1}_2 \otimes E^{(3)}_k. \end{cases} \tag{1.32}$$

First, we consider the case of the second order term x^2:

$$\begin{aligned} {}^\forall x \in \mathbb{R}^4,\ \ (x^\dagger \otimes \mathbf{1}_k)C^\dagger C(x \otimes \mathbf{1}_k) = \mathbf{1}_2 \otimes E^{(1)}_k(x^2) &\iff {}^\forall x^\mu, x^\nu \in \mathbb{R},\ \ x^\mu x^\nu (e^\dagger_\mu \otimes \mathbf{1}_k)C^\dagger C(e_\nu \otimes \mathbf{1}_k) = \alpha x^2 \mathbf{1}_2 \otimes \tilde{E}^{(1)}_k \\ &\iff (e^\dagger_\mu \otimes \mathbf{1}_k)C^\dagger C(e_\nu \otimes \mathbf{1}_k) = \left(\alpha\delta_{\mu\nu}\mathbf{1}_2 + B_{\mu\nu}\right) \otimes \tilde{E}^{(1)}_k, \end{aligned} \tag{1.33}$$

where $B_{\mu\nu}$ is an arbitrary anti-symmetric tensor $B_{\mu\nu} = -B_{\nu\mu}$ and $\alpha \in \mathbb{C}$. What in the form of $C^\dagger C$ that satisfy (1.33)? We give this answer in the following lemma.

Lemma 1.2.1.

$$(e^\dagger_\mu \otimes \mathbf{1}_k)C^\dagger C(e_\nu \otimes \mathbf{1}_k) = \left(\alpha\delta_{\mu\nu}\mathbf{1}_M + B_{\mu\nu}\right) \otimes \tilde{E}^{(1)}_k \iff C^\dagger C = \mathbf{1}_2 \otimes \tilde{E}^{(1)}_k. \tag{1.34}$$

Proof. $\Leftarrow$).
Suppose that $C^\dagger C = \mathbf{1}_2 \otimes \tilde{E}^{(1)}_k$, then

$$(e^\dagger_\mu \otimes \mathbf{1}_k)\mathbf{1}_2 \otimes \tilde{E}^{(1)}_k(e_\mu \otimes \mathbf{1}_k) = e^\dagger_\mu e_\nu \otimes \tilde{E}^{(1)}_k = \left(\delta_{\mu\nu}\mathbf{1}_2 + \eta^{(+)}_{\mu\nu}/2\right) \otimes \tilde{E}^{(1)}_k. \tag{1.35}$$

Recall that $\eta^{(+)}_{\mu\nu}$ is the anti-symmetric tensor by the definition (1.11), hence this assertion has proven.

$\Rightarrow$).
Since the r.h.s. is written by the tensor product, the l.h.s. also have to decompose into the tensor product: $C^\dagger C = X_2 \otimes Y_k$. Now X is 2×2-matrix and Y_k is $k \times k$-matrix. Hence the l.h.s. on the tensor product becomes

$$e^\dagger_\mu X_2 e_\nu \propto \alpha\delta_{\mu\nu}\mathbf{1}_2 + B_{\mu\nu}. \tag{1.36}$$

For the assumption $B_{\mu\nu} = -B_{\nu\mu}$, we have $e^\dagger_\nu X_2 e_\mu \propto \alpha\delta_{\nu\mu}\mathbf{1}_2 + A_{\nu\mu} = \alpha\delta_{\mu\nu}\mathbf{1}_2 - B_{\mu\nu}$. Therefore we obtain the following relation:

$$e^\dagger_\mu X_2 e_\nu + e^\dagger_\nu X_2 e_\mu \propto \delta_{\mu\nu}\mathbf{1}_2. \tag{1.37}$$

Now we recall that $e^\dagger_\mu e_\nu + e^\dagger_\nu e_\mu = 2\delta_{\mu\nu}\mathbf{1}_2$, thus $X_2 = \beta\mathbf{1}_2$ ($\beta \in \mathbb{C}$). Therefore

$$C^\dagger C = \mathbf{1}_2 \otimes \tilde{Y}_k, \tag{1.38}$$

where $\tilde{Y}_k := \beta Y_k$. Substitute this result into the l.h.s. of the right equation in (1.34), and we obtain $\tilde{Y}_k = \tilde{E}^{(1)}_k$. $\mathcal{Q.E.D.}$

Next we consider the case of the first order term x^1.

$$\begin{aligned}
&{}^\forall x \in \mathbb{R}^4,\ \ (x^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(x \otimes \mathbf{1}_k) = \mathbf{1}_2 \otimes 2E_k^{(2)}(x) \\
\iff\ & {}^\forall x^\mu \in \mathbb{R},\ \ x^\mu \left((e_\mu^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(e_\mu \otimes \mathbf{1}_k)\right) = x^\mu \mathbf{1}_2 \otimes 2\tilde{E}_{k,\mu}^{(2)} \\
\iff\ & (e_\mu^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(e_\mu \otimes \mathbf{1}_k) = \mathbf{1}_2 \otimes 2\tilde{E}_{k,\mu}^{(2)},
\end{aligned} \tag{1.39}$$

where we define $\tilde{E}_{k,\mu}^{(2)}$ by $E_k^{(2)}(x) = x^\mu \tilde{E}_{k,\mu}^{(2)}$. The conditiont that $C^\dagger D$ satisfies (1.39) is given by the following lemma.

Lemma 1.2.2. *Let Y^μ be $k \times k$ Hermite matrix, then*

$$(e_\mu^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(e_\mu \otimes \mathbf{1}_k) = \mathbf{1}_2 \otimes 2\tilde{E}_{k,\mu}^{(2)} \iff C^\dagger D = e_\mu \otimes Y^\mu, \tag{1.40}$$

where $Y^\mu = \delta^{\mu\nu}\tilde{E}_{k,\nu}^{(2)}$.

Proof. $\Leftarrow$).
It can be easily shown by substituting the r.h.s. equation in (1.40) into the l.h.s. equation.

$$(e_\mu^\dagger \otimes \mathbf{1}_k)(e_\nu \otimes Y^\nu) + (e_\nu^\dagger \otimes Y^{\dagger\ \nu})(e_\mu \otimes \mathbf{1}_k) = e_\mu^\dagger e_\nu \otimes Y^\nu + e_\nu^\dagger e_\mu \otimes Y^{\dagger\ \nu} = \left(e_\mu^\dagger e_\nu + e_\nu^\dagger e_\mu\right) \otimes Y^\nu = 2\delta_{\mu\nu}\mathbf{1}_2 \otimes Y^\nu. \tag{1.41}$$

$\Rightarrow$).
Because of the same reason for the proof of lemma 1.2.2, $C^\dagger D$ have to decompose to 2×2-matrix X and $k \times k$-matrix Y: $C^\dagger D = X \otimes Y$. Now Y is arbitrary complex matrix $Y \in \mathrm{GL}(k;\mathbb{C})$, thus Y is classified to any of the Hermite, anti-Hermite, or non-Hermite. We consider the each case of Y in the following.

First, let Y be non-Hermite then

$$(e_\mu^\dagger \otimes \mathbf{1}_k)X \otimes Y + X^\dagger \otimes Y^\dagger(e_\mu \otimes \mathbf{1}_k) = e_\mu^\dagger X \otimes Y + X^\dagger e_\mu \otimes Y^\dagger. \tag{1.42}$$

This result accords with $\mathbf{1}_2 \otimes \tilde{E}_{k,\mu}^{(2)}$ if and only if the cases of $X = 0$ or $Y = 0$, however this assumption is not suitable. Therefore we reject the assumption that Y is non-Hermite.

Next, let Y be Hermite then

$$(e_\mu^\dagger \otimes \mathbf{1}_k)X \otimes Y + X^\dagger \otimes Y^\dagger(e_\mu \otimes \mathbf{1}_k) = e_\mu^\dagger X \otimes Y + X^\dagger e_\mu \otimes Y = (e_\mu^\dagger X + X^\dagger e_\mu) \otimes Y. \tag{1.43}$$

This result accords with $\mathbf{1}_2 \otimes \tilde{E}_{k,\mu}^{(2)}$ if and only if $X = e_\nu,\ Y = Y^\nu$.

Finally, let Y be anti-Hermite then

$$(e_\mu^\dagger \otimes \mathbf{1}_k)X \otimes Y + X^\dagger \otimes Y^\dagger(e_\mu \otimes \mathbf{1}_k) = e_\mu^\dagger X \otimes Y - X^\dagger e_\mu \otimes Y = (e_\mu^\dagger X - X^\dagger e_\mu) \otimes Y \tag{1.44}$$

This result accords with $\mathbf{1}_2 \otimes \tilde{E}_{k,\mu}^{(2)}$ if and only if $X = ie_\nu,\ Y = Y^\nu$. Now, we replace the imaginary unit on X for on Y ($ie_\nu \mapsto e_\nu$ and $Y^\nu \mapsto iY^\nu$), then anti-Hermite case is essentially same as the Hermite case.

Therefore we have the result that $C^\dagger D = e_\mu \otimes Y^\mu$ and Y^ν is Hermite. Finally, compare this result with the l.h.s. equation, and we obtain $Y^\mu = \delta^{\mu\nu}\tilde{E}_{k,\nu}^{(2)}$. $\mathcal{Q.E.D.}$

Finally, we consider the case of the zero order term x^0, but this is most simplify form already.
Therefore we decompose the ADHM constraint (1.25) to three x-independent conditions:

$$C^\dagger C = \mathbf{1}_2 \otimes \tilde{E}_k^{(1)}, \tag{1.45a}$$
$$C^\dagger D = e_\mu \otimes \tilde{E}_{k,\mu}^{(2)}, \tag{1.45b}$$
$$D^\dagger D = \mathbf{1}_2 \otimes \tilde{E}_l^{(3)}, \tag{1.45c}$$

where $\tilde{E}_{k,\mu}^{(2)}$ is a Hermite matrix and $E_k^{(1)} = x^2\tilde{E}_k^{(1)} + 2x^\mu\tilde{E}_{k,\mu}^{(2)} + E_k^{(3)}$.

It is well-known that the ADHM data is more simplified by using the gauge freedom on ADHM data without loss of generality [21]. The ADHM constraint (1.25), the Weyl equation (1.26), and the zero-mode nomarization $V^\dagger V = \mathbf{1}_2$ are invariant under the following transformation:

$$C \mapsto C' = \mathcal{U}C\mathcal{R}, \quad D \mapsto \mathcal{U}D\mathcal{R}, \quad V \mapsto V' = \mathcal{U}V, \tag{1.46}$$

where $\mathcal{U} \in \mathrm{U}(2+2k)$ and $\mathcal{R} = \mathbf{1}_2 \otimes \mathcal{R}_k \in \mathbf{1}_2 \otimes \mathrm{GL}(k;\mathbb{C})$. Using this $\mathrm{U}(2+2k) \times \mathrm{GL}(k;\mathbb{C})$ transformation, we can fix the ADHM data to the so-called the canonical form:

$$C = \begin{pmatrix} 0_{[2]\times[2k]} \\ \mathbf{1}_{2k} \end{pmatrix}_{[2+2k]\times[2k]}, \quad D = \begin{pmatrix} S_{[2]\times[2k]} \\ T_{[2k]\times[2k]} \end{pmatrix}. \tag{1.47}$$

Here the matrix subscript $[a]\times[b]$ means the matrix size, and the symbol $S_{[2]\times[2k]}$ stands for $\begin{pmatrix} S_{1\,[2]\times[k]} & S_{2[2]\times[k]} \end{pmatrix}$. In the canonical form, the informations of the all ADHM data are included in the matrices S and T. Let us now rewrite the x-independent condition for the ADHM constraint (1.45) in the canonical form. In this case, $C^\dagger C = \mathbf{1}_{2k} = \mathbf{1}_2 \otimes \mathbf{1}_k$, thus the condition (1.45a) is automatically satisfied. The condition (1.45b) means that the matrix $C^\dagger D$ is written with e_μ. In the canonical form, $C^\dagger D = T$ thus (1.45b) becomes

$$T = e_\mu \otimes T^\mu, \tag{1.48}$$

where T^μ is a $k \times k$ Hermite matrix. The condition (1.45c) is rewritten as

$$S^\dagger S + T^\dagger T = \mathbf{1}_2 \otimes E_k^{(3)}. \tag{1.49}$$

$S^\dagger S$ is decomposed with e_μ:

$$S^\dagger S = \begin{pmatrix} S_1^\dagger S_1 & S_1^\dagger S_2 \\ S_2^\dagger S_1 & S_2^\dagger S_2 \end{pmatrix} =: e_\mu \otimes I^\mu = \begin{pmatrix} I^4 - iI^3 & -I^2 - iI^1 \\ I^2 - I^1 & I^4 + iI^3 \end{pmatrix}, \tag{1.50}$$

where we defined I^μ as

$$\begin{aligned} I^1 &:= \frac{i}{2}(S_2^\dagger S_1 + S_1^\dagger S_1), \qquad I^2 := \frac{1}{2}(S_2^\dagger S_1 - S_1^\dagger S_2), \\ I^3 &:= \frac{i}{2}(S_1^\dagger S_1 - S_2^\dagger S_2), \qquad I^4 := \frac{1}{2}(S_1^\dagger S_1 + S_2^\dagger S_2). \end{aligned} \tag{1.51}$$

Now we recall that $e_\mu := \delta_{4\mu}\mathbf{1}_2 - i\delta_{\mu i}\sigma_i$ and $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$, (1.49) becomes

$$\begin{aligned} & S^\dagger S + T^\dagger T = \mathbf{1}_2 \otimes I^4 - i\sigma_i \otimes I^i + \mathbf{1}_2 \otimes T^\mu T^\mu + i\sigma_i \otimes \left(\frac{1}{2}\varepsilon_{ijk}[T^j, T^k] + [T^i, T^4]\right) = \mathbf{1}_2 \otimes E_k^{(3)} \\ \Longleftrightarrow\ & i\sigma_i \otimes \left(\frac{1}{2}\varepsilon_{ijk}[T^j, T^k] + [T^i, T^4] - I^i\right) + \mathbf{1}_2 \otimes (T^\mu T^\mu + I^4) = \mathbf{1}_2 \otimes E_k^{(3)} \\ \Longleftrightarrow\ & \frac{1}{2}\varepsilon_{ijk}[T^j, T^k] + [T^i, T^4] - I^i = 0. \end{aligned} \tag{1.52}$$

This equation is usually called the ADHM equation. Therefore the ADHM constraint in the canonical form becomes the simplified equation (1.52) and the condition of ADHM data (1.48). Note that there are residual symmetries that leave the canonical form (1.47) invariant. The transformations are given by

$$S_\alpha \mapsto S'_\alpha = QS_\alpha R, \quad T^\mu \mapsto T'^\mu = R^\dagger T^\mu R, \tag{1.53}$$

where the index $\alpha = 1, 2$, $Q \in \mathrm{SU}(2)$ and $R \in \mathrm{U}(k)$. We sometimes call this residual symmetry as the gauge freedom of the ADHM data.

The transformation of the zero mode that preserves the normalization condition $V^\dagger V = \mathbf{1}_2$ is given by

$$V(x) \mapsto V(x)g(x), \quad g(x) \in \mathrm{U}(2). \tag{1.54}$$

Note that this transformation is independent of the transformation (1.46). This zero mode transformation leads to a gauge field transformation through (1.27). Indeed the gauge field is transformed into

$$A_\mu \mapsto g^{-1}(x)A_\mu g(x) + g^{-1}(x)\partial_\mu g(x). \tag{1.55}$$

This transformation is same as the ordinary gauge transformation. Therefore the instantons that are produced from the ADHM construction possess the unitary goup U(2). Hence, we note that the ADHM construction does not impose the speciality condition on the gauge group in general, namely the gauge group is not the special unitary group SU(2). We can decompose the group U(2) into the special group SU(2) part and U(1) part: $U(2) = SU(2) \ltimes U(1)$. Here the symbol $\ltimes$ means the semidirect product of the group. Therefore we usually must fix the element U(1) by hand when we consider SU(2) instanton in the ADHM construction.

There is a useful formula, known as the Osborn's formula, to calculate the action density from the ADHM data. The Osborn's formula [22, 23] is

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu} = -\partial^2 \partial^2 \ln \det E_k^{-1}. \tag{1.56}$$

It can be easily shown that the index k denotes instanton number with using this formula. Using ASD equation and $E_k^{-1}(\infty) = x^{-2}\mathbf{1}_k$,

$$\begin{aligned} Q &= -\frac{1}{8\pi^2}\frac{1}{2}\int d^4x \,\mathrm{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right] = \frac{1}{16\pi^2}\int d^4x\, \partial^2\partial^2 \ln\det E_k^{-1} \\ &= \frac{1}{16\pi^2}\int dS_x^\mu \partial_\mu \partial^2 \mathrm{Tr}_k \ln E_k^{-1} = \frac{8}{16\pi^2}\int d\Omega_3 \mathrm{Tr}_k \mathbf{1}_k = k. \end{aligned} \tag{1.57}$$

Here $\int d\Omega_3$ means the three-dimensional spherical integration.

1.2.1 Some ADHM data with U(2)

In U(2) gauge group, various ADHM data have been constructed in previous studies. In this subsection, we show some well known ADHM data with U(2) gauge group.

BPST instanton

In the $k = 1$, the ADHM data in the canonical form is the simplest one:

$$C = \begin{pmatrix} 0 \\ \mathbf{1}_2 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda \mathbf{1}_2 \\ -a^\mu e_\mu \end{pmatrix}, \tag{1.58}$$

where $\lambda \in \mathbb{R}$ is the size modulus and $a^\mu \in \mathbb{R}$ is the position modulus of the instanton. The solution to the Weyl equation (1.26) is

$$V = \frac{1}{\sqrt{\rho}} \begin{pmatrix} \tilde{x}^\dagger \\ -\lambda \mathbf{1}_2 \end{pmatrix}, \tag{1.59}$$

where $\tilde{x}^\dagger = (x^\mu - a^\mu) e_\mu^\dagger$ and $\rho = \lambda^2 + ||\tilde{x}||^2$. We easily confirm that this ADHM data reproduce the BPST instanton gauge field (1.14) by using (1.27).

't Hooft k-instanton

The 't Hooft ADHM data [7] is given by

$$T^\mu = \mathrm{diag}_{p=1}^k(-a_p^\mu), \quad S = \mathbf{1}_2 \otimes \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \end{pmatrix}, \tag{1.60}$$

with $a_p^\mu \in \mathbb{R}$ is the instanton position and $\lambda_p \in \mathbb{R}$ is the instanton size moduli respectively. The solution to the Weyl equation that is associated with (1.60) is

$$V = \frac{1}{\sqrt{\phi}} \begin{pmatrix} -\mathbf{1}_2 \\ e_\mu \otimes \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu}{||\tilde{x}_p||^2} S^\dagger \right), \end{pmatrix} \tag{1.61}$$

where $\phi \coloneqq 1 + \sum_{p=1}^k \frac{\lambda_p^2}{||\tilde{x}_p||^2}$, $\tilde{x}_p^\mu \coloneqq x^\mu - a_p^\mu$ and $||\tilde{x}_p||^2 \coloneqq \tilde{x}_p^\mu \tilde{x}_p^\mu$ (p is not summed).

JNR k-instanton

The JNR ADHM data [24] is

$$\Delta_{(4)} = \begin{pmatrix} \mathbf{1}_2 \otimes \Lambda \\ \mathbf{1}_2 \otimes \mathbf{1}_k \end{pmatrix} \cdot x \otimes \mathbf{1}_k + \begin{pmatrix} -a_0 \otimes \Lambda \\ \mathrm{diag}_{p=1}^k(-a_p) \end{pmatrix} = \begin{pmatrix} \tilde{x}_0 \otimes \Lambda \\ \tilde{X} \end{pmatrix} = e_\mu \otimes \begin{pmatrix} \tilde{x}_0^\mu \Lambda \\ \mathrm{diag}_{p=1}^k(\tilde{x}_p^\mu) \end{pmatrix}, \tag{1.62}$$

where $\Lambda = \begin{pmatrix} \lambda_1/\lambda_0 & \ldots & \lambda_k/\lambda_0 \end{pmatrix}$, $\tilde{x}_p = (x^\mu - a_p^\mu)e_\mu$, $a_p \coloneqq a_p^\mu e_\mu$, and $\tilde{X} = \mathrm{diag}_{p=1}^k(\tilde{x}_p)$. Here $\lambda_p \in \mathbb{R}$ and $a_p^\mu \in \mathbb{R}$ are moduli paremeters. The JNR ADHM data is not in the canonical form and contain more moduli parameters than the 't Hooft one. The latter is obtained from the former by the limit $a_0 \to \infty$ and $\lambda \to \infty$ with fixed $a_0/\lambda = 1$, thus, in this sentence, the JNR ADHM data is a generalization of the 't Hooft ADHM data. The solution to the Weyl equation is

$$V = \frac{1}{\sqrt{\phi}} \begin{pmatrix} -\mathbf{1}_2 \\ \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p}{\|\tilde{x}_p\|^2} \cdot \tilde{x}_0^\dagger \otimes {}^t\Lambda \right), \end{pmatrix} \tag{1.63}$$

where $\phi \coloneqq 1 + \frac{\|\tilde{x}_0\|^2}{\lambda_0^2} \sum_{p=1}^k \left(\frac{\lambda_p^2}{\|\tilde{x}_p\|^2} \right)$.

Chapter 2

In $4n$ dimensions ($n \geq 2$)

In this chapter, let us consider higher-dimensional instantons and an ADHM construction of these instantons. The discussion of this chapter is based on [25, 26].

Naturally, we consider generalization of the four-dimensional instantons by generalizing the ASD equation to higher dimensions. There are several kinds of "instantons" in higher dimensions, and these have been studied in various contexts. One of the main types of instantons is sometimes called a secular type instanton [27, 28]. The secular type instantons are the solutions to the linear equation $F_{\mu\nu} = \lambda T_{\mu\nu\rho\sigma} F^{\rho\sigma}, \lambda \neq 0, (\mu, \nu, \rho, \sigma = 1, \dots, d)$. Here $d > 4$ and the symbol $T_{\mu\nu\rho\sigma}$ is an anti-symmetric constant tensor that respects subgroups of the SO(d) Lorentz group. However, the Chern numbers that are associated with the secular type instantons are not finite and quantized in general. In this sense, the secular type instantons are not topological solitons. An ADHM construction of the secular type instantons in $4n$ ($n = 1, 2, 3, \dots$) dimensions has been studied in [29].

On the other hand, we can consider the another type equation which is the straightforward generalization of the four-dimensional ASD equation: $F(n) = \pm *_{4n} F(n)$. Here $F(n)$ is the nth wedge products of the field strength 2-form F. This equation is called the $4n$-dimensional ASD equation and solutions to this equation are called ASD instantons. One of the most important characters of the ASD instantons is that these topological charges, which are defined by the $2n$-th Chern number $c_{2n} = \frac{1}{(2n)!(2\pi)^{2n}} \int \mathrm{Tr} F(2n)$, are finite and quantized when the homotopy group is non-trivial: $\pi_{4n-1}(G) \neq 0$. This type instanton was firstly studied by Tchrakian [30], [1] and he constructed a spherical symmetry SO($4n$) instanton in $4n$ dimensions which was generalization of the four-dimensional Belavin-Polyakov-Schwartz-Tyupkin(BPST) instanton. Furthermore, in $4n$ dimensions, an axially symmetric SO($4n$) one-instanton was presented explicitly in [32]. This instanton is the analogy of the axially symmetric Witten solution in four dimensions. The existence of axially symmetric SO($4n$) multi-instantons has been proved analytically in [33, 34]. However, in $4n$ ($n \geq 2$) dimensions, a SO($4n$) instanton of which symmetry less than axially symmetry does not exist [35]. For this reason, we propose the following question. Can we construct higher-dimensional instantons of which other gauge groups and symmetries? We will consider an approach of a higher-dimensional ADHM construction to elucidate this question. In this paper, we treat only the case of which the base manifold is Euclidean space $\mathbb{R}^{4n}$. Note that there are the ASD instantons on other base manifolds also, for instance, the case of complex projective space $\mathbb{C}P^m$ was discussed in [36].

In this chapter, we mainly consider an ADHM construction of the $4n$-dimensional ($n \geq 2$) ASD instantons with the unitary gauge group U(N). The first non-trivial case ($n = 2$), the eight-dimensional ADHM construction, has been studied in [25]. The $4n$-dimensional ADHM construction is generalization of the eight-dimensional one. We will show that this is a general scheme to construct the ASD instantons and the known one-instantons in $4n$ dimensions can be reproduced from this scheme [2]. Moreover, we will discuss higher-dimensional multi-instantons by introducing specific ADHM data which solve ADHM constraints, and we mention calorons and the monopole limit in higher dimensions.

The organization of this chapter in as follows. Section 2.1 is about an ASD type instantons in $4n$ dimensions. In this section, we introduce a generalized Yang-Mills action which leads the 4n-dimensional ASD equation and a $4n$-dimensional BPST one instanton. Moreover we discuss a generalization of the 't Hooft symbol in four dimensions, called as the $4n$-dimensional ASD tensor which plays central rules of the following discussions. Section 2.2 is discussed an $4n$-dimensional ADHM construction of ASD instantons with U(N) gauge group. We find that, in higher dimensions, there are two type ADHM constraints on the ADHM data. The one is straightforward generalization of the four-dimensionnal one, while the another, essentially a new constraint type, comes from the non-linearity of the higher dimensional ASD equations. Section 2.3 is about higher-dimensional ADHM data with U(2^{2n-1}) gauge group. We consider the generalization of the four-dimensional ADHM data, namely an BPST type, a 't

[1] The special case of $n = 2$ was studied independently in [31].

[2] Note that the gauge group of this reproduced one-instanton expand to the unitary group U(2^{2n-1}).

Hooft type and a JNR type data. However we show that these multi-instanton ADHM data are well-defined only if we assume that the each single-instanton well-separated. In section 2.4, we show some calculations in more detail. Subsection 2.4.1 is about the difference of the formulations about the Hermite or anti-Hermite gauge in higher dimensions. Subsection 2.4.2 is about the higher-dimensional ASD tensor. In this subsection, we prove some properties, which is used in this paper, of the ASD tensor. In subsection 2.4.3, we prove the existence of the inverse matrix of $E_k^{(a)}$ which is appeared in the section 2.2. Subsection 2.4.4. is detailed calculation that leads an approximate charge density of 't Hooft $k = 2, 3$ instantons. Section 2.5 is about the Clifford algebra and the 4n-dimensional ASD tensor. The 4n-dimensional ASD tensor which is introduced in section 2.1 is constructed from the Clifford algebra. In this section, we discuss the properties of the Clifford algebra and give the explicit matrix representations of the Clifford algebra namely the ASD tensor. Section 2.6 is discussed about the properties of a higher-dimensional ADHM equations, and we explicitly lead a eight-dimensional ADHM equations with U(8) gauge group. Section 2.7 is about calorons, sometimes called as periodic instantons , in higher dimensions. It is known that the Harrington-Shepard caloron in four dimensions is leaded by the 't Hooft multi-instantons that are periodic in one of the four coordinates. We consider the same method to apply in higher dimensions.

2.1 (Anti-)self-dual instantons in $4n$ dimensions

In this section, we study ASD instantons in $4n$-dimensional Euclidean space with the flat metric. The $4n$-dimensional ASD equation is defined as the generalization of the usual four-dimensional ASD equation (1.3):

$$F(n) = \pm *_{4n} F(n), \tag{2.1}$$

where $*_{4n}$ is the $4n$-dimensional Hodge dual operator, $F(n) = F \wedge \cdots \wedge F$ (n times) and $F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is the gauge field strength 2-form of which component is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Here A_μ is the anti-Hermite gauge field ($A_\mu^\dagger = -A_\mu$) which takes value in a Lie algebra $\mathcal{G}$. The Lie algebra $\mathcal{G}$ is associated with the non-Abelian gauge group G and the greek indices $\mu, \nu, \cdots = 1, 2, \ldots, 4n$ are the $4n$-dimensional Euclidean space indices. The component expression of the ASD equation (2.1) is

$$F_{[\mu_1\mu_2} \ldots F_{\mu_{2n-1}\mu_{2n}]} = \pm \frac{1}{(2n)!} \varepsilon_{\mu_1\mu_2\ldots\mu_{2n-1}\mu_{2n}\nu_1\nu_2\ldots\nu_{2n-1}\nu_{2n}} F_{\nu_1\nu_2} \ldots F_{\nu_{2n-1}\nu_{2n}}, \tag{2.2}$$

where $\varepsilon_{\mu_1\mu_2\ldots\mu_{2n-1}\mu_{2n}\nu_1\nu_2\ldots\nu_{2n-1}\nu_{2n}}$ is the anti-symmetric tensor in $4n$ dimensions and the bracket $[\mu_1\mu_2 \ldots \mu_{2n}]$ means the anti-symmetrization of indices with the weight $1/(2n)!$. The $4n$-dimensional ASD instantons are defined as the solutions to the $4n$-dimensional ASD equations (2.2). As in four dimensions, the square of the $4n$-dimensional Hodge dual operator on Euclidean space becomes the identity operator: $*_{4n}^2 = 1$. Hence the $2n$-form linear dual tangent space $\Lambda^{2n}(T_x^*\mathbb{R}^{4n})$ is able to decompose with the eigenspace that the eigenvalue is ± 1:

$$\Lambda^{2n}(T_x^*\mathbb{R}^{4n}) = \Lambda_+^{2n}(T_x^*\mathbb{R}^{4n}) \oplus \Lambda_-^{2n}(T_x^*\mathbb{R}^{4n}). \tag{2.3}$$

where we have defined $\Lambda_\pm^{2n}(T_x^*\mathbb{R}^{4n}) := \left\{\omega \in \Lambda^{2n}(T_x^*\mathbb{R}^{4n}) \mid *_{4n}\, \omega = \pm\omega\right\}$. Therefore we can always classify the $2n$-form as the self dual or the anti-self dual. This fact guarantees that the $4n$-dimensional instantons are able to be classified as self dual instantons or anti-self dual instantons.

The action that gives the $4n$-dimensional ASD equation (2.1) is given by

$$S = (-1)^n \mathcal{N}_n \int \mathrm{Tr}\left[F(n) \wedge *_{4n} F(n)\right]. \tag{2.4}$$

We call this action as the generalized Yang-Mills action. Here $\mathcal{N}_n$ is the normalization constant in $4n$ dimensions which will be determined on the last in this section. If we choose the Hermite gauge field ($A_\mu^\dagger = A_\mu$) then the action coefficient signature is replaced 1 instead of $(-1)^n$ (see subsection 2.4.1). We easily show that the Bogomol'nyi completion of the action is

$$S = (-1)^n \frac{\mathcal{N}_n}{2} \int \mathrm{Tr}\left[(F(n) \mp *_{4n} F(n))^2 \pm 2F(2n)\right] \geq \pm(-1)^n \mathcal{N}_n \int \mathrm{Tr} F(2n), \tag{2.5}$$

where we have defined

$$(F(n) \mp *_{4n} F(n))^2 = (F(n) \mp *_{4n} F(n)) \wedge *_{4n} (F(n) \mp *_{4n} F(n)). \tag{2.6}$$

Therefore the Bogomol'nyi bound of the action (2.4) is saturated when the solutions satisfy the $4n$-dimensional ASD equation (2.1), and then the action is bounded from below by the $2n$-th Chern number $S = \pm(-1)^n \mathcal{N}_n \int \mathrm{Tr} F(2n)$.

The $4n$-dimensional Belavin-Polyakov-Schwartz-Tyupkin(BPST) type instanton was discussed in [30, 31, 37]. We review this type instanton in the following. The gauge field of the BPST type instanton is

$$A_\mu(x) = -\frac{1}{2}\frac{\tilde{x}^\nu}{\lambda^2 + ||\tilde{x}||^2}\Sigma^{(\pm)}_{\mu\nu}, \tag{2.7}$$

where we have defined $\tilde{x}^\mu = x^\mu - a^\mu$, $a^\mu \in \mathbb{R}$ is the position of the instanton, $\lambda \in \mathbb{R}$ is the instanton size and $||\tilde{x}||^2 = (x^\mu - a^\mu)(x_\mu - a_\mu)$. The symbol $\Sigma^{(\pm)}_{\mu\nu}$ is a $4n$-dimensional ASD tensor, and this is an analogy of the 't Hooft symbol in four dimensions.

The ASD tensor in $4n$ dimensions is given by

$$\Sigma^{(+)}_{\mu\nu} = e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu, \ \ \Sigma^{(-)}_{\mu\nu} = e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu, \tag{2.8}$$

with $e_i, e^\dagger_i$, which we call the ASD basis in $4n$ dimensions, are defined by

$$e_\mu = \delta_{\mu\#}\mathbf{1}_{2^{2n-1}} + \delta_{\mu i}\Gamma^{(-)}_i, \ \ e^\dagger_\mu = \delta_{\mu\#}\mathbf{1}_{2^{2n-1}} + \delta_{\mu i}\Gamma^{(+)}_i, \quad (i = 1, \ldots, 4n-1, \ \ \# = 4n), \tag{2.9}$$

where $\Gamma^{(\pm)}_i$ are $2^{2n-1} \times 2^{2n-1}$ matrices that satisfy the relation $\{\Gamma^{(\pm)}_i, \Gamma^{(\pm)}_j\} = -2\delta_{ij}\mathbf{1}_{2^{2n-1}}$, and $\mathbf{1}_{2^{2n-1}}$ is the identity matrix. The element $\Gamma^{(\pm)}_i$ is defined by $\Gamma^{(\pm)}_i = \frac{1}{2}(1 \pm \omega)\Gamma_i$ and we choose $\Gamma^{(\pm)}_i$ that satisfies the relation: $\Gamma^{(+)}_i = -\Gamma^{(-)}_i$. Here Γ_i is the matrix representation of the $(4n-1)$-dimensional complex Clifford algebra: $\Gamma_i \in C\ell_{4n-1}(\mathbb{C})$, and ω is the chirality element which is defined by

$$\omega = (-1)^{n+1}\Gamma_1\Gamma_2\ldots\Gamma_{4n-1}. \tag{2.10}$$

The explicit matrix representation of the $(4n-1)$-dimensional complex Clifford algebras can be found in subsection 2.5. The ASD basis e_μ is the generalization of the quaternion basis in four dimensions and is normalized as $\mathrm{Tr}\left[e_\mu e^\dagger_\nu\right] = 2^{2n-1}\delta_{\mu\nu}$. The relation of convenient for calculations is

$$e_\mu e^\dagger_\nu + e_\nu e^\dagger_\mu = e^\dagger_\mu e_\nu + e^\dagger_\nu e_\mu = 2\delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}. \tag{2.11}$$

The ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$ satisfies the $4n$-dimensional ASD relation:

$$\Sigma^{(\pm)}_{[\mu_1\mu_2}\ldots\Sigma^{(\pm)}_{\mu_{2n-1}\mu_{2n}]} = \pm\frac{1}{(2n)!}\varepsilon_{\mu_1\mu_2\ldots\mu_{2n-1}\mu_{2n}\nu_1\nu_2\ldots\nu_{2n-1}\nu_{2n}}\Sigma^{(\pm)}_{\nu_1\nu_2}\ldots\Sigma^{(\pm)}_{\nu_{2n-1}\nu_{2n}}, \tag{2.12}$$

where the upper script sign of $\Sigma^{(\pm)}_{\mu\nu}$ corresponds to the sign in the r.h.s. of (2.12). For later convenience, we calculate the following quantities:

$$\Sigma^{(\pm)}_{\mu_1\mu_2}\ldots\Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} = \varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}\Sigma^{(\pm)}_{12}\ldots\Sigma^{(\pm)}_{(4n-1)(4n)} = \pm(-1)^n 2^{2n}\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}\mathbf{1}_{2^{2n-1}}. \tag{2.13}$$

These results are proved at subsubsection 2.4.2 in p.30.

The field strength $F_{\mu\nu}$ of the BPST type instantons is evaluated to be

$$F_{\mu\nu} = \frac{\lambda^2}{(\lambda^2 + ||\tilde{x}||^2)^2}\Sigma^{(\pm)}_{\mu\nu}. \tag{2.14}$$

Then the field strength (2.14) manifestly satisfies the $4n$-dimensional ASD equation (2.2) by using (2.12). The ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$ satisfies the commutation relation:

$$\left[\Sigma^{(\pm)}_{\mu\nu}, \Sigma^{(\pm)}_{\rho\sigma}\right] = 4\left(\delta_{\nu\rho}\Sigma^{(\pm)}_{\mu\sigma} - \delta_{\nu\sigma}\Sigma^{(\pm)}_{\mu\rho} + \delta_{\mu\rho}\Sigma^{(\pm)}_{\sigma\nu} - \delta_{\mu\sigma}\Sigma^{(\pm)}_{\rho\nu}\right). \tag{2.15}$$

Hence, we find that $\Sigma^{(\pm)}_{\mu\nu}$ is the spinor-representation of the SO($4n$) Lie algebra. Therefore the gauge group of the $4n$-dimensional BPST type instanton is the special orthogonal group SO($4n$) and its homotopy group is $\pi_{4n-1}(\mathrm{SO}(4n)) = \mathbb{Z} \oplus \mathbb{Z}$. Note that it is sufficient that the homotopy group contains at least one $\mathbb{Z}$ factor to classify instantons by the integer topological charge.

Next, we determine the normalization constant $\mathcal{N}_n$. This is defined by the condition that the topological charge of the BPST instanton (2.14) becomes one. The topological charge Q of the $4n$-dimensional instantons is defined by the $2n$-th Chern number:

$$Q = (-1)^n\mathcal{N}_n\int_{\mathbb{R}^{4n}}\mathrm{Tr}F(2n) = (-1)^n\mathcal{N}_n\int_{\mathbb{R}^{4n}}d^{4n}x\mathrm{Tr}\left[\left(\frac{1}{2}\right)^{2n}\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}F_{\mu_1\mu_2}\ldots F_{\mu_{4n-1}\mu_{4n}}\right]. \tag{2.16}$$

We easily calculate the topological charge of the BPST type instantons (2.14) by using (2.13). The result is

$$\begin{aligned}
Q &= (-1)^n \mathcal{N}_n \frac{1}{2^{2n}} \int_{\mathbb{R}^{4n}} d^{4n}x \left(\frac{\lambda^2}{(\lambda^2 + \|\tilde{x}\|^2)^2} \right)^{2n} \mathrm{Tr}\left[\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}} \Sigma^{(\pm)}_{\mu_1\mu_2} \ldots \Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} \right] \\
&= \mathcal{N}_n \int_{\mathbb{R}^{4n}} d^{4n}x \left(\frac{\lambda^2}{(\lambda^2 + \|\tilde{x}\|^2)^2} \right)^{2n} \cdot \pm(4n)!2^{2n-1} \\
&= \pm(4n)!2^{2n-1} \mathcal{N}_n \int_{S^{4n-1}} d\Omega_{4n-1} \int_0^\infty dr \left(\frac{1}{(1+r^2)^2} \right)^{2n} \\
&= \pm \frac{2^{2n}(4n)!n\pi^{2n}}{\Gamma(2n+1)} B(2n, 2n) \mathcal{N}_n = \pm(2n)!(2\pi)^{2n} \mathcal{N}_n,
\end{aligned} \tag{2.17}$$

where $B(2n, 2n)$ is the beta function and we have used the following relations:

$$\int_{S^{m-1}} d\Omega_{m-1} = \frac{m\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)} r^{m-1}, \tag{2.18a}$$

$$\begin{aligned}
\int_0^\infty dr \frac{r^{4n-1}}{(1+r^2)^{4n}} &= \int_0^{\pi/2} \cos^{8n}\theta \tan^{4n-1}\theta \cdot \cos^{-2}\theta \, d\theta \quad \because).r = \tan\theta \\
&= \int_0^{\pi/2} \cos^{4n-1}\theta \sin^{4n-1}\theta \, d\theta = \frac{1}{2} B(2n, 2n),
\end{aligned} \tag{2.18b}$$

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.18c}$$

We define that the topological charge of the instantons is a positive number when the instantons satisfy the self-dual equation, i.e. the plus sign in (2.2). Therefore the $4n$-dimensional normalization constants $\mathcal{N}_n$ is determined to be

$$\mathcal{N}_n = \frac{1}{(2n)!(2\pi)^{2n}}. \tag{2.19}$$

2.2 U(N) ADHM construction in $4n$ dimensions ($n \geq 2$)

In this section, we study an ADHM construction of the ASD instanton in the $4n$-dimensional Euclidean space with the flat metric. In the following, we choose the anti-self-dual solutions to the equation (2.2) and we use the matrix representation of the Clifford algebra $C\ell_{4n-1}(\mathbb{C})$. This explicit form can be found in subsection 2.5. We first introduce the $4n$-dimensional Weyl operator:

$$\Delta = C(x \otimes \mathbf{1}_k) + D, \tag{2.20}$$

where $x = x^\mu e_\mu$, the symbol $\otimes$ means the tensor product, C and D are $(N + 2^{2n-1}k) \times 2^{2n-1}k$ constant matrices which are called the ADHM data, and N corresponds to the rank of the unitary group, we will show this fact for later. If we consider self-dual solutions then we must choose the basis $e^\dagger_\mu$ instead of e_μ. In the next section, we will show that the integer k corresponds to the instanton number which is defined by the $2n$-th Chern number $k = |\mathcal{N}_n \int \mathrm{Tr}F(2n)|$. Now we demand that the Weyl operator satisfies the first ADHM constraint:

$$\Delta^\dagger \Delta = \mathbf{1}_{2^{2n-1}} \otimes E^{(1)}_k, \tag{2.21}$$

and the second ADHM constraint:

$$C^\dagger \Delta \left(\Delta^\dagger \Delta\right)^{-1} \Delta^\dagger C = \mathbf{1}_{2^{2n-1}} \otimes E^{(2)}_k, \tag{2.22}$$

where $\Delta^\dagger$ is the Hermitian conjugate matrix of Δ, $E^{(1)}_k$ and $E^{(2)}_k$ are invertible $k \times k$ matrices. The first ADHM constraint (2.21) is the natural generalization of the four-dimensional one [7]. On the other hand, the second ADHM constraint (2.22) is the analogy of the eight-dimensional one [25]. In addition, the Weyl operator requires the non-degeneracy condition: rank $\Delta = 2^{2n-1}k$, and the existence of the inverse $E^{(a)}_k$ ($a = 1, 2$) is guaranteed by this condition (see subsubsection 2.4.3). Here rank A means the rank of the matrix A, and the non-degeneracy condition of the Weyl operator is satisfied if and only if the ADHM data C, D satisfy the condition: rank C = rank $D = 2^{2n-1}k$. For later convenience, let us analyze the ADHM constraints in more detail. For (2.20), the first ADHM constraint (2.21) becomes

$$(x^\dagger \otimes \mathbf{1}_k)C^\dagger C(x \otimes \mathbf{1}_k) + (x^\dagger \otimes \mathbf{1}_k)C^\dagger D + D^\dagger C(x \otimes \mathbf{1}_k) + D^\dagger D = \mathbf{1}_{2^{2n-1}} \otimes E^{(1)}_k(x). \tag{2.23}$$

The ADHM constraints hold for all $x \in \mathbb{R}^{4n}$, hence we can decompose the first ADHM constraint to three x-independent conditions:

$$C^\dagger C = \mathbf{1}_{2^{2n-1}} \otimes E_k^{(1,1)}, \tag{2.24a}$$

$$C^\dagger D = e_\mu \otimes E_{k,\mu}^{(1,2)}, \tag{2.24b}$$

$$D^\dagger D = \mathbf{1}_{2^{2n-1}} \otimes E_k^{(1,3)}, \tag{2.24c}$$

where $E_{k,\mu}^{(1,2)}$ is a Hermite matrix and $E_k^{(1)} = x^2 E_k^{(1,1)} + 2x^\mu E_{k,\mu}^{(1,2)} + E_k^{(1,3)}$. These calculations are guaranteed by the lemma 1.2.1 and 1.2.2, because the properties of ASD basis which are only used in these proofs are the general results that are derived from Clifford algebra namely these lemma to be prove true in higher dimensions. Similarly, the second ADHM constraint (2.22) expands to

$$\begin{aligned} &C^\dagger C(x \otimes \mathbf{1}_k)(\mathbf{1}_{2^{2n-1}} \otimes f)(x^\dagger \otimes \mathbf{1}_k)C^\dagger C + C^\dagger C(x \otimes \mathbf{1}_k)(\mathbf{1}_{2^{2n-1}} \otimes f)D^\dagger C \\ &\quad + C^\dagger D(\mathbf{1}_{2n-1} \otimes f)(x^\dagger \otimes \mathbf{1}_k)C^\dagger C + C^\dagger D(\mathbf{1}_{2^{2n-1}} \otimes f)D^\dagger C = \mathbf{1}_{2^{2n-1}} \otimes E_k^{(2)}(x), \end{aligned} \tag{2.25}$$

where $f^{-1} = E_k^{(1)}$. When we discuss the x-independent conditions of the second ADHM constraint, we can ignore the x that is included in f because the matrix f is already placed in the r.h.s. for the tensor product. For (2.24a) and $x^\dagger x = xx^\dagger = x^2 \mathbf{1}_{2^{2n-1}}$, the x^2 term in (2.25) automatically satisfies the constraint. The x^1 terms in (2.25) expands to

$$\begin{aligned} &C^\dagger C(x \otimes \mathbf{1}_k)(\mathbf{1}_{2^{2n-1}} \otimes f)D^\dagger C + C^\dagger D(\mathbf{1}_{2n-1} \otimes f)(x^\dagger \otimes \mathbf{1}_k)C^\dagger C \\ &\quad = x^\nu \left(e_\nu e_\mu^\dagger \otimes E_k^{(1,1)} f E_{k,\mu}^{(1,2)} + e_\mu e_\nu^\dagger \otimes E_{k,\mu}^{(1,2)} f E_k^{(1,1)} \right). \end{aligned} \tag{2.26}$$

These terms satisfy the constraint for all x if and only if the following condition holds:

$$E_k^{(1,1)} f E_{k,\mu}^{(1,2)} = E_{k,\mu}^{(1,2)} f E_k^{(1,1)}. \tag{2.27}$$

Next we consider the x^0 term in (2.25). This term becomes

$$C^\dagger D(\mathbf{1}_{2^{2n-1}} \otimes f)D^\dagger C = \left(\delta_{\mu\nu} \mathbf{1}_{2^{2n-1}} + \Sigma_{\mu\nu}^{(-)}/2 \right) \otimes E_{k,\mu}^{(1,2)} f E_{k,\nu}^{(1,2)}, \tag{2.28}$$

here we have used $e_\mu e_\nu^\dagger = \delta_{\mu\nu} \mathbf{1}_{2^{2n-1}} + \Sigma_{\mu\nu}^{(-)}/2$. For this equation, we obtain the following condition for the x^0 term:

$$\Sigma_{\mu\nu}^{(-)} \otimes E_{k,\mu}^{(1,2)} f E_{k,\nu}^{(1,2)} = 0. \tag{2.29}$$

Therefore we obtained the two x-independent conditions of the second ADHM constraint, namely (2.27) and (2.29).

Let us show that how to obtain the gauge field of the anti-self-dual instanton from the ADHM data. Following the ADHM construction in four dimensions [7], we first consider zero modes of the Weyl operator Δ. The null-space of the Hermitian conjugate matrix $\Delta^\dagger$ is N-dimensional, as it has N fewer rows than columns. The basis vectors for this null-space can be assembled into an $(N + 2^{2n-1}k) \times N$ matrix $V(x)$, which is sometimes called the zero mode. This fact means that the zero mode $V(x)$ is the solution to the Weyl equation:

$$\Delta^\dagger V(x) = 0, \tag{2.30}$$

and the zero mode $V(x)$ is normalized as $V^\dagger V = \mathbf{1}_N$. The zero mode V and the Weyl operator Δ satisfy the following relation which is called the completeness relation:

$$\mathbf{1}_{N+2^{2n-1}k} - VV^\dagger = \Delta(\Delta^\dagger \Delta)^{-1} \Delta^\dagger. \tag{2.31}$$

We can easily prove this relation by using a $(N + 2^{2n-1}k) \times (N + 2^{2n-1}k)$ matrix $W = \begin{pmatrix} \Delta & V \end{pmatrix}$. Because of the non-degeneracy condition, the Weyl equation (2.30) and the normalization: $V^\dagger V = \mathbf{1}_N$, the columns of W are linearly independent. Therefore the matrix W is invertible, and the following equation is an identity equation: $W(W^\dagger W)^{-1} W^\dagger = \mathbf{1}_{N+2^{2n-1}k}$. We can obtain the completeness relation by expanding the l.h.s. term $W(W^\dagger W)^{-1} W^\dagger$ with Δ and V. We employ the ansatz of the gauge field $A_\mu(x)$ is given by the pure gauge form:

$$A_\mu(x) = V^\dagger(x) \partial_\mu V(x). \tag{2.32}$$

Next we confirm that the field strength $F_{\mu\nu}$ from the ansatz (2.32) automatically satisfies the anti-self-dual equation (2.2). For the Weyl equation (2.30) and the completeness relation (2.31), the field strength becomes

$$F_{\mu\nu} = V^\dagger C \left(e_\mu \otimes \mathbf{1}_k \right) \left(\Delta^\dagger \Delta \right)^{-1} \left(e_\nu^\dagger \otimes \mathbf{1}_k \right) C^\dagger V - (\mu \leftrightarrow \nu). \tag{2.33}$$

Now we use the first ADHM constraint (2.21) then the factor $(\Delta^\dagger\Delta)^{-1}$ commutes with the basis $e_\mu \otimes \mathbf{1}_k$. Hence the field strength becomes

$$F_{\mu\nu} = V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{\mu\nu} \otimes \mathbf{1}_k\right)C^\dagger V. \tag{2.34}$$

Therefore the multi-product of the field strengths is

$$F_{\mu_1\mu_2}\ldots F_{\mu_{2n-1}\mu_{2n}} = \left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{\mu_1\mu_2} \otimes \mathbf{1}_k\right)C^\dagger V\right)\ldots\left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{\mu_{2n-1}\mu_{2n}} \otimes \mathbf{1}_k\right)C^\dagger V\right). \tag{2.35}$$

We order that $\Sigma^{(-)}_{\mu\nu} \otimes \mathbf{1}_k$ commute with $C^\dagger VV^\dagger C$ in (2.35), thus we demand the following condition:

$$e_\mu \otimes \mathbf{1}_k\left(C^\dagger VV^\dagger C\right) = \left(C^\dagger VV^\dagger C\right)e_\mu \otimes \mathbf{1}_k. \tag{2.36}$$

Use the completeness relation (2.31) then the condition (2.36) is decomposed as

$$e_\mu \otimes \mathbf{1}_k\left(C^\dagger C\right) = \left(C^\dagger C\right)e_\mu \otimes \mathbf{1}_k, \quad e_\mu \otimes \mathbf{1}_k\left(C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C\right) = \left(C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C\right)e_\mu \otimes \mathbf{1}_k. \tag{2.37}$$

For (2.24a), the first condition is automatically satisfied when the first ADHM constraint (2.21) holds. On the other hand, the second condition is just the second ADHM constraint (2.22). We find that the condition (2.36) is satisfied when the first and second ADHM constraints hold. Therefore, for the condition (2.36), the multi-product of the field strengths becomes

$$F_{\mu_1\mu_2}\ldots F_{\mu_{2n-1}\mu_{2n}} = V^\dagger C\left(\Delta^\dagger\Delta\right)^{-1}\left(\Sigma^{(-)}_{\mu_1\mu_2}\ldots\Sigma^{(-)}_{\mu_{2n-1}\mu_{2n}} \otimes \mathbf{1}_k\right)C^\dagger V\left(V^\dagger C\left(\Delta^\dagger\Delta\right)^{-1}C^\dagger V\right)^{n-1}. \tag{2.38}$$

Since $\Sigma^{(-)}_{\mu_1\mu_2}\ldots\Sigma^{(-)}_{\mu_{2n-1}\mu_{2n}}$ satisfies the anti-self-dual relation (2.12), we have shown that the field strengths $F_{\mu\nu}$ that are constructed from the $4n$-dimensional ADHM construction satisfy the anti-self-dual equation in $4n$ dimensions (2.2).

We show that the ADHM data can transform more simplify form without loss of generality. It is easy to find that the Weyl equation (2.30), the normalization condition $V^\dagger V = \mathbf{1}_N$, the first and second ADHM constraints (2.21),(2.22) are invariant under the following transformations:

$$C \mapsto C' = \mathcal{U}C\mathcal{R}, \quad D \mapsto D' = \mathcal{U}D\mathcal{R}, \quad V \mapsto V' = \mathcal{U}V, \tag{2.39}$$

where $\mathcal{U} \in \mathrm{U}(N + 2^{2n-1}k)$ and $\mathcal{R} = \mathbf{1}_{2^{2n-1}} \otimes \mathcal{R}_k \in \mathbf{1}_{2^{2n-1}} \otimes \mathrm{GL}(k;\mathbb{C})$. Using this $\mathrm{U}(N + 2^{2n-1}k) \times \mathrm{GL}(k;\mathbb{C})$ transformation, we can fix the ADHM data to the so-called a canonical form:

$$C = \begin{pmatrix} 0_{[N]\times[2^{2n-1}k]} \\ \mathbf{1}_{2^{2n-1}k} \end{pmatrix}, \quad D = \begin{pmatrix} S_{[N]\times[2^{2n-1}k]} \\ T_{[2^{2n-1}k]\times[2^{2n-1}k]} \end{pmatrix}. \tag{2.40}$$

Here the matrix subscript $[a]\times[b]$ means the matrix size, and the symbol $S_{[N]\times[2^{2n-1}k]}$ stands for $\begin{pmatrix} S_{1\ [N]\times[k]} & \ldots & S_{2^{2n-1}\ [N]\times[k]} \end{pmatrix}$. The existence of the canonical form is guaranteed by the non-degeneracy condition. In the canonical form, all the ADHM data are included in the matrices S and T. Let us now rewrite the x-independent conditions of the first and second ADHM constraints (2.24),(2.27),(2.29) in the canonical form. In this case, $C^\dagger C = \mathbf{1}_{2^{2n-1}k} = \mathbf{1}_{2^{2n-1}} \otimes \mathbf{1}_k$ thus the condition (2.24a) is automatically satisfied. The condition (2.24b) means that the matrix $C^\dagger D$ is written with the ASD basis e_μ. In the canonical form, $C^\dagger D = T$ thus (2.24b) becomes

$$T = e_\mu \otimes T^\mu, \tag{2.41}$$

where T^μ is a $k \times k$ Hermite matrix. The condition (2.24c) is rewritten as

$$S^\dagger S + T^\dagger T = \mathbf{1}_{2^{2n-1}} \otimes E^{(1,3)}_k. \tag{2.42}$$

This x-independent condition is the natural generalization of four-dimensional one which is usually called the ADHM equation. On the other hand, the x-independent conditions of the second ADHM constraint lead to new type ADHM equations. In the canonical form, $E^{(1,1)}_k = \mathbf{1}_k$ and $\delta^{\mu\nu}E^{(1,2)}_{k,\nu} = T^\mu$ thus the condition (2.27) becomes

$$fT^\mu = T^\mu f. \tag{2.43}$$

For this condition, the condition (2.29) is rewritten as

$$\Sigma^{(-)}_{\mu\nu} \otimes T^\mu T^\nu = 0. \tag{2.44}$$

In higher dimensions, the ADHM data must satisfy the new type ADHM equations (2.43) and (2.44) in addition to the standard type one (2.42). Finally, we note that there are residual symmetries which leave the canonical form (2.40) invariant. The transformations are given by

$$S_a \mapsto S'_a = QS_aR, \quad T^\mu \mapsto T'^\mu = R^\dagger T^\mu R, \tag{2.45}$$

where the index a runs from 1 to 2^{2n-1}, $Q \in \mathrm{SU}(N)$ and $R \in \mathrm{U}(k)$.

Next, we study the gauge group of the instantons that are generated from the ADHM construction. The transformation of the zero mode which preserves the normalization condition $V^\dagger V = \mathbf{1}_N$ is given by

$$V(x) \mapsto V(x)g(x), \quad g(x) \in \mathrm{U}(N). \tag{2.46}$$

Note that this transformation is independent of the transformation (2.39). This zero mode transformation leads to a gauge field transformation through (2.32). Indeed, the gauge field is transformed to

$$A_\mu \mapsto g^{-1}(x)A_\mu g(x) + g^{-1}(x)\partial_\mu g(x). \tag{2.47}$$

This transformation is same as the ordinary gauge transformation. Hence, the instantons that are generated from the ADHM construction possess the unitary gauge group $\mathrm{U}(N)$. Because of this fact, in the special case $k = 0$, we find that the ansatz (2.32) gives a pure gauge, namely, it automatically solves the ASD equation (2.2) in the vacuum sector. We are interested in instantons that are characterized by the instanton number k, but the homotopy groups become trivial when the rank of the unitary group is small. Therefore the rank of the gauge group N is restricted by the condition that the homotopy group $\pi_{4n-1}(\mathrm{U}(N))$ is non-trivial. The non-trivial homotopy groups of the (special) unitary group are

$$\pi_{4n-1}(\mathrm{U}(N)) = \pi_{4n-1}(\mathrm{SU}(N)) = \mathbb{Z}, \quad N \geq 2n. \tag{2.48}$$

For this reason, we demand the condition $N \geq 2n$ when we consider the topological instantons. In addition, we note that the ADHM construction does not impose the speciality condition on the gauge group in general, namely the gauge group is not the special unitary group $\mathrm{SU}(N)$. We can decompose the group $\mathrm{U}(N)$ into the special group $\mathrm{SU}(N)$ part and $\mathrm{U}(1)$ part: $\mathrm{U}(N) = \mathrm{SU}(N) \ltimes \mathrm{U}(1)$. Here the symbol $\ltimes$ means the semidirect product of the group. Usually, we must fix the element of $\mathrm{U}(1)$ by hand when we consider $\mathrm{SU}(N)$ instantons in the ADHM construction.

In higher dimensions, although we still do not know that a formula that similar to the Osborn's formula (1.56) which is given the relation between the action and the ADHM data directly, we can show a formula that alternative to the Osborn's formula to calculate the action from the ADHM data. Recall the topological charge Q for the $4n$-dimensional instantons is defined by the $2n$-th Chern number $Q = (-1)^n \mathcal{N}_n \int \mathrm{Tr}F(2n)$. For (2.38),

$$\begin{aligned}
F(2n) &= \left(\frac{1}{2}\right)^{2n} F_{\mu_1\mu_2} \ldots F_{\mu_{4n-1}\mu_{4n}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{4n-1}} \wedge dx^{\mu_{4n}} \\
&= \frac{1}{2^{2n}} \varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}} \left(V^\dagger C \left(\Delta^\dagger\Delta\right)^{-1} \left(\left(\Sigma^{(\pm)}_{\mu_1\mu_2} \ldots \Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}}\right) \otimes \mathbf{1}_k\right) C^\dagger V\right) \left(V^\dagger C \left(\Delta^\dagger\Delta\right)^{-1} C^\dagger V\right)^{2n-1} d^{4n}x \\
&= \frac{1}{2^{2n}} \underbrace{\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}} \varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}}_{=(4n)!} \left(V^\dagger C \left(\Delta^\dagger\Delta\right)^{-1} \left(\pm(-1)^n 2^{2n} \mathbf{1}_{2^{2n-1}} \otimes \mathbf{1}_k\right) C^\dagger V\right) \left(V^\dagger C \left(\Delta^\dagger\Delta\right)^{-1} C^\dagger V\right)^{2n-1} d^{4n}x \\
&= \pm(-1)^n (4n)! \left(V^\dagger C \left(\Delta^\dagger\Delta\right)^{-1} C^\dagger V\right)^{2n} d^{4n}x.
\end{aligned} \tag{2.49}$$

Here we used (2.13). Now we define the charge density $\mathcal{Q}$ as $Q = \mathcal{N}_n \int d^{4n}x\mathcal{Q}$, then we obtain the charge density formula:

$$\mathcal{Q} = \pm(4n)!\mathrm{Tr}\left(V^\dagger C(\Delta^\dagger\Delta)^{-1}C^\dagger V\right)^{2n}, \tag{2.50}$$

where $\pm$ corresponds to the ASD solution (tensor) respectively. When the ADHM data is the canonical form (2.40), we can rewrite (2.50) as more simplify. Since the circulation law of trace, (2.50) becomes

$$\begin{aligned}
\mathcal{Q} &= \pm(4n)!\mathrm{Tr}_N \left(V^\dagger C(\Delta^\dagger\Delta)^{-1}C^\dagger V\right)^{2n} \\
&= \pm(4n)!\mathrm{Tr}_{2^{2n-1}k} \left((\Delta^\dagger\Delta)^{-1}C^\dagger VV^\dagger C\right)^{2n} \\
&= \pm(4n)!\mathrm{Tr}_{2^{2n-1}k} \left((\Delta^\dagger\Delta)^{-1}C^\dagger(1 - \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger)C\right)^{2n} \\
&= \pm(4n)!\mathrm{Tr}_{2^{2n-1}k} \left((\Delta^\dagger\Delta)^{-1}(1 - C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C)\right)^{2n} \\
&= \pm(4n)!\mathrm{Tr}_{2^{2n-1}k} \left[\mathbf{1}_{2^{2n-1}} \otimes \left(E_k^{-1}(\mathbf{1}_k - E_k^{(2)})\right)\right]^{2n}.
\end{aligned} \tag{2.51}$$

Use $\mathrm{Tr}_{ab}\left[\mathbf{1}_a \otimes X_{[b]}\right] = a\mathrm{Tr}_b X_{[b]}$, then we obtain the fomula with canonical form:

$$Q = \pm 2^{2n-1}(4n)!\mathrm{Tr}_k\left(\left(E_k^{(1)}\right)^{-1}\left(\mathbf{1}_k - E_k^{(2)}\right)\right)^{2n}. \tag{2.52}$$

Now we have introduced the ADHM construction of the ASD instantons in $4n$ dimensions. Here we have some comments on the higher dimensional ADHM construction. Compared with the four-dimensional ADHM construction, the first ADHM constraints (2.21) is the natural generalization of the four-dimensional one. On the other hand, the second ADHM constraint (2.22) is an essentially new constraint and this new constraint corresponds to the non-linearity of the ASD equation (2.2). It is difficult to construct the multi-instantons in higher dimensions, because of the non-linearity of the ASD equations. In the higher dimensional ADHM construction is similar to this situation, namely the constructions of the multi-instantons are difficult by the second ADHM constraint. We will discuss this fact in more detail in the next section.

2.3 Higher-dimensional ADHM data with U(2^{2n-1}) gauge group

In this section, we introduce explicit ADHM data in higher dimensions ($n \geq 2$). However, it is hard that we find an essentially new ADHM data, hence we will consider the data type that is generalizing the well known four-dimensional one and choose the rank of the gauge group to $N = 2^{2n-1}$. Here we recall that the first ADHM constraint is the natural generalization of the four-dimensional one. Therefore the data type that is generalizing the four-dimensional ADHM data already satisfies the first ADHM constraint, and we call this data type as an ADHM "ansatz".

The second ADHM constraint (2.22) contains the inverse matrix $(\Delta^\dagger\Delta)^{-1}$, hence the calculation of this constraint is hard in general. Therefore we use the following constraint instead of the second ADHM constraint to confirm that the ADHM ansatz is well-defined as a higher-dimensional ADHM data:

$$C^\dagger V V^\dagger C = \mathbf{1}_{2^{2n-1}} \otimes E_k^{(3)}, \tag{2.53}$$

where $E_k^{(3)}$ is an invertible $k \times k$ matrix. The existence of the inverse $E_k^{(3)}$ is guaranteed by that $E_k^{(1)}$ and $E_k^{(2)}$ are invertible. Although this constraint contains the zero mode V, the calculation of the Weyl equation (2.30) is more easily than the calculation of the inverse matrix $(\Delta^\dagger\Delta)^{-1}$ in general, namely we can calculate the constraint (2.53) more easily than the second ADHM constraint. This constraint is same as the condition (2.36), therefore the ADHM ansatz satisfy the first ADHM constraint then this constraint is same as the second ADHM constraint for (2.37).

2.3.1 BPST type one-instanton

In the case of $k = 1$, the ADHM ansatz in the canonical form is the simplest one:

$$C = \begin{pmatrix} 0 \\ \mathbf{1}_{2^{2n-1}} \end{pmatrix}, \quad D = \begin{pmatrix} \lambda \mathbf{1}_{2^{2n-1}} \\ -a^\mu e_\mu \end{pmatrix}, \tag{2.54}$$

where $\lambda \in \mathbb{R}$ is the size modulus and $a^\mu \in \mathbb{R}$ is the position modulus of the instanton. The solution to the Weyl equation (2.30) is

$$V = \frac{1}{\sqrt{\rho}}\begin{pmatrix} \tilde{x}^\dagger \\ -\lambda \mathbf{1}_{2^{2n-1}} \end{pmatrix}, \tag{2.55}$$

where $\tilde{x}^\dagger = (x^\mu - a^\mu)e_\mu^\dagger$ and $\rho = \lambda^2 + ||\tilde{x}||^2$. The l.h.s. of the constraint (2.53) that is associated with the BPST type ADHM ansatz (2.54) is proportional to the identity $\mathbf{1}_{2^{2n-1}}$:

$$C^\dagger V V^\dagger C = \frac{\lambda^2}{\rho}\mathbf{1}_{2^{2n-1}}. \tag{2.56}$$

Hence, this ansatz (2.54) is well-defined as the ADHM data of the anti-self-dual one-instanton. Indeed, we easily confirm that this ADHM data reproduces the BPST type gauge field (2.7) by using (2.32).

Let us now recalculation the topological charge with using the formula (2.52).

$$\Delta^\dagger\Delta = \mathbf{1}_{2^{2n-1}} \otimes \left(\lambda^2 + ||\tilde{x}||^2\right), \tag{2.57}$$

and

$$C^\dagger\Delta\left(\Delta^\dagger\Delta\right)\Delta^\dagger C = \tilde{x}\left(\mathbf{1}_{2^{2n-1}} \otimes \left(\lambda^2 + ||\tilde{x}||^2\right)^{-1}\right)\tilde{x}^\dagger = \mathbf{1}_{2^{2n-1}} \otimes \frac{||\tilde{x}||^2}{\lambda^2 + ||\tilde{x}||^2}. \tag{2.58}$$

Here the ADHM data (2.54) is the anti-instanton case, thus we chose the minus sigh in (2.52):

$$Q = -2^{2n-1}(4n)!\left(\frac{1}{\lambda^2+||\tilde{x}||^2}\left(1-\frac{||\tilde{x}||^2}{\lambda^2+||\tilde{x}||^2}\right)\right)^{2n} = -2^{2n-1}(4n)!\left(\frac{\lambda^2}{(\lambda^2+||\tilde{x}||^2)^2}\right)^{2n}. \tag{2.59}$$

This is just the BPST instanton's charge density (ref.(2.17)), thus we reproduce $Q=-1$.

2.3.2 't Hooft type ansatz

We next consider the ADHM data with higher charges. A natural candidate for multi-instanton ADHM data is generalization of the four-dimensional 't Hooft one (1.60). However, in the case of $n \geq 2$, it was shown that the simple generalization of the 't Hooft type ADHM ansatz is not well-defined as the higher-dimensional ADHM data in [25]. In the following, we will show this fact in more detail. The 't Hooft type ADHM ansatz is given by

$$T^\mu = \mathrm{diag}_{p=1}^k\left(-a_p^\mu\right), \qquad S = \mathbf{1}_{2^{2n-1}}\otimes\begin{pmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_k\end{pmatrix}, \tag{2.60}$$

with $a_p^\mu \in \mathbb{R}$ is the instanton position and $\lambda_p \in \mathbb{R}$ is the instanton size moduli respectively. The Weyl operator that is associated with the 't Hooft type ADHM ansatz is

$$\Delta^\dagger = \begin{pmatrix}S^\dagger & e_\mu^\dagger\otimes(x^\mu\mathbf{1}_k+T^\mu)\end{pmatrix}. \tag{2.61}$$

Then we find

$$\Delta^\dagger\Delta = S^\dagger S + e_\mu^\dagger e_\nu\otimes\mathrm{diag}_{p=1}^k\left(\tilde{x}_p^\mu\tilde{x}_p^\nu\right) = \mathbf{1}_{2^{2n-1}}\otimes\begin{pmatrix}\lambda_1^2+||\tilde{x}_1||^2 & \lambda_1\lambda_2 & \ldots & \lambda_1\lambda_k\\ \lambda_2\lambda_1 & \lambda_2^2+||\tilde{x}_2||^2 & \ldots & \lambda_2\lambda_k\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_k\lambda_1 & \lambda_k\lambda_2 & \ldots & \lambda_k^2+||\tilde{x}_k||^2\end{pmatrix}, \tag{2.62}$$

where we have defined $\tilde{x}_p^\mu \coloneqq x^\mu - a_p^\mu$ and $||\tilde{x}_p||^2 = \tilde{x}_p^\mu\tilde{x}_p^\mu$ (p is not summed). Therefore the ADHM ansatz (2.60) satisfies the first ADHM constraint (2.21).

The solution to the Weyl equation (2.30) is

$$V = \frac{1}{\sqrt{\phi}}\begin{pmatrix}-\mathbf{1}_{2^{2n-1}}\\ \left(e_\mu\otimes\mathrm{diag}_{p=1}^k\left(\frac{\tilde{x}_p^\mu}{||\tilde{x}_p||^2}\right)\right)S^\dagger\end{pmatrix}. \tag{2.63}$$

Here $\phi = 1+\sum_{p=1}^k\frac{\lambda_p^2}{||\tilde{x}_p||^2}$. We then examine the constraint (2.53). We first calculate $C^\dagger V,\ V^\dagger C$:

$$C^\dagger V = e_\mu\otimes\frac{1}{\sqrt{\phi}}\mathrm{diag}_{p=1}^k\left(\frac{\tilde{x}_p^\mu}{||\tilde{x}_p||^2}\right)S^\dagger = e_\mu\otimes\frac{1}{\sqrt{\phi}}\begin{pmatrix}\lambda_1X_1^\mu\\ \lambda_2X_2^\mu\\ \vdots\\ \lambda_kX_k^\mu\end{pmatrix}, \tag{2.64a}$$

$$V^\dagger C = \left(C^\dagger V\right)^\dagger = e_\mu^\dagger\otimes\frac{1}{\sqrt{\phi}}S\,\mathrm{diag}_{p=1}^k\left(\frac{\tilde{x}_p^\mu}{||\tilde{x}_p||^2}\right) = e_\mu^\dagger\otimes\frac{1}{\sqrt{\phi}}\begin{pmatrix}\lambda_1X_1^\mu & \lambda_2X_2^\mu & \ldots & \lambda_kX_k^\mu\end{pmatrix}. \tag{2.64b}$$

Thus plugging the zero-mode (2.63) into $C^\dagger VV^\dagger C$, we have

$$C^\dagger VV^\dagger C = \left(\delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}+\Sigma_{\mu\nu}^{(-)}/2\right)\otimes\frac{1}{\phi}E^{\mu\nu}_{(\text{'t Hooft})}, \tag{2.65}$$

where

$$E^{\mu\nu}_{(\text{'t Hooft})} = \begin{pmatrix}\lambda_1^2X_1^\mu X_1^\nu & \lambda_1\lambda_2X_1^\mu X_2^\mu & \ldots & \lambda_1\lambda_kX_1^\mu X_k^\nu\\ \lambda_2\lambda_1X_2^\mu X_1^\nu & \lambda_2^2X_2^\mu X_2^\nu & \ldots & \lambda_2\lambda_kX_2^\mu X_k^\nu\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_k\lambda_1X_k^\mu X_1^\nu & \lambda_k\lambda_2X_k^\mu X_2^\nu & \ldots & \lambda_k^2X_k^\mu X_k^\nu\end{pmatrix}. \tag{2.66}$$

Here we have used the relation $e_\mu e_\nu^\dagger = \delta_{\mu\nu}\mathbf{1}_{2^{2n-1}} + \Sigma_{\mu\nu}^{(-)}/2$ and defined $X_p^\mu = \tilde{x}_p^\mu/||\tilde{x}_p||^2$. Since the constraint (2.53) requires that the r.h.s. of (2.65) is proportional to $\mathbf{1}_{2^{2n-1}}$, we have the following conditions on the moduli λ_a and a_m^μ:

$$\lambda_m\lambda_n\left(x^\mu - a_m^\mu\right)(x^\nu - a_n^\nu)\,\Sigma_{\mu\nu}^{(-)} = 0, \tag{2.67}$$

where the indices m, n run from 1 to k and are not summed. This condition is trivially satisfied in the case of $k = 1$, however, for arbitrary moduli parameters λ_m, a_m, this is not satisfied in the higher charges $k \geq 2$. Therefore the simple generalization of the 't Hooft type ADHM ansatz is not well-defined as the ADHM data with higher charges ($k \geq 2$).

Let us now demand the following condition for moduli parameters to satisfy the condition (2.67):

$$||a_m^\mu - a_n^\mu||^2 \gg \lambda_m\lambda_n, \tag{2.68}$$

for all m and n. This condition means that each instanton is well-separated, hence we call this condition as the well-separated limit or the dilute instanton gas limit (approximation) [21]. In the well-separated limit (2.68), we neglect all the off-diagonal components of the matrix $E_k^{(1)}$ in (2.21):

$$E_k^{(1)} = \begin{pmatrix} \lambda_1^2 + ||\tilde{x}_1||^2 & \dots & \lambda_1\lambda_k \\ \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 & \dots & \lambda_k^2 + ||\tilde{x}_k||^2 \end{pmatrix} \simeq \begin{pmatrix} \lambda_1^2 + ||\tilde{x}_1||^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k^2 + ||\tilde{x}_k||^2 \end{pmatrix}. \tag{2.69}$$

Thus the second ADHM constraint becomes

$$C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C = e_\mu^\dagger \otimes \mathrm{diag}_{p=1}^k \tilde{x}_p^\mu \left(\mathbf{1}_{2^{2n-1}} \otimes \left(E_k^{(1)}\right)^{-1}\right) e_\nu \otimes \mathrm{diag}_{p=1}^k \tilde{x}_p^\nu \simeq \mathbf{1}_{2^{2n-1}} \otimes \mathrm{diag}_{p=1}^k \left[\frac{||\tilde{x}_p||^2}{\lambda_p^2 + ||\tilde{x}_p||^2}\right]. \tag{2.70}$$

Therefore the 't Hooft type ADHM ansatz (2.60) in the well-separated limit satisfies the second ADHM constraint (2.22).

We proceed to evaluate the instanton charge for the 't Hooft type ADHM data as follows. The 't Hooft type ADHM ansatz does not satisfy the second ADHM constraint (2.22), thus this ansatz is not the ADHM data in strictly speaking. Although it is not exact that we use the formula (2.50) which it is assumed that the ADHM ansatz satisfy the second ADHM constraint, we consider that this formula give an approximate charge density if each instanton is well-separated. Plugging (2.62) and (2.64) into the formula (2.50):

$$\begin{aligned} Q \simeq{}& -(4n)!\mathrm{Tr}\Bigg\{e_\mu^\dagger \otimes \frac{1}{\sqrt{\phi}}\begin{pmatrix}\lambda_1 X_1^\mu & \lambda_2 X_2^\mu & \dots & \lambda_k X_k^\mu\end{pmatrix} \\ & \left[\mathbf{1}_{2^{2n-1}} \otimes \begin{pmatrix} \lambda_1^2 + ||\tilde{x}_1||^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_k \\ \lambda_1\lambda_2 & \lambda_2^2 + ||\tilde{x}_2||^2 & \dots & \lambda_2\lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\lambda_k & \lambda_2\lambda_k & \dots & \lambda_k^2 + ||\tilde{x}_k||^2 \end{pmatrix}^{-1}\right] e_\mu \otimes \frac{1}{\sqrt{\phi}}\begin{pmatrix}\lambda_1 X_1^\mu \\ \lambda_2 X_2^\mu \\ \vdots \\ \lambda_k X_k^\mu\end{pmatrix}\Bigg\}^{2n} \\ ={}& \frac{-(4n)!}{\phi^{2n}}\mathrm{Tr}\left\{e_\mu^\dagger e_\nu \otimes \left(\begin{pmatrix}\frac{\lambda_1\tilde{x}_1^\mu}{||\tilde{x}_1||^2} & \frac{\lambda_2\tilde{x}_2^\mu}{||\tilde{x}_2||^2} & \dots & \frac{\lambda_k\tilde{x}_k^\mu}{||\tilde{x}_k||^2}\end{pmatrix}\begin{pmatrix} \lambda_1^2 + ||\tilde{x}_1||^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_k \\ \lambda_1\lambda_2 & \lambda_2^2 + ||\tilde{x}_2||^2 & \dots & \lambda_2\lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\lambda_k & \lambda_2\lambda_k & \dots & \lambda_k^2 + ||\tilde{x}_k||^2 \end{pmatrix}^{-1}\begin{pmatrix}\frac{\lambda_1\tilde{x}_1^\mu}{||\tilde{x}_1||^2} \\ \frac{\lambda_2\tilde{x}_2^\mu}{||\tilde{x}_2||^2} \\ \vdots \\ \frac{\lambda_k\tilde{x}_k^\mu}{||\tilde{x}_k||^2}\end{pmatrix}\right)\right\}^{2n}. \end{aligned} \tag{2.71}$$

Therefore we obtain the approximate charge density of the 't Hooft k-instanton:

$$Q \simeq \frac{-(4n)!}{\phi^{2n}}\mathrm{Tr}\left\{f^{\mu\nu}e_\mu^\dagger e_\nu\right\}^{2n}, \tag{2.72}$$

where the scalor $f^{\mu\nu}$ is defined as

$$f^{\mu\nu} = \begin{pmatrix}\frac{\lambda_1\tilde{x}_1^\mu}{||\tilde{x}_1||^2} & \frac{\lambda_2\tilde{x}_2^\mu}{||\tilde{x}_2||^2} & \dots & \frac{\lambda_k\tilde{x}_k^\mu}{||\tilde{x}_k||^2}\end{pmatrix}\begin{pmatrix} \lambda_1^2 + ||\tilde{x}_1||^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_k \\ \lambda_1\lambda_2 & \lambda_2^2 + ||\tilde{x}_2||^2 & \dots & \lambda_2\lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\lambda_k & \lambda_2\lambda_k & \dots & \lambda_k^2 + ||\tilde{x}_k||^2 \end{pmatrix}^{-1}\begin{pmatrix}\frac{\lambda_1\tilde{x}_1^\nu}{||\tilde{x}_1||^2} \\ \frac{\lambda_2\tilde{x}_2^\nu}{||\tilde{x}_2||^2} \\ \vdots \\ \frac{\lambda_k\tilde{x}_k^\nu}{||\tilde{x}_k||^2}\end{pmatrix}. \tag{2.73}$$

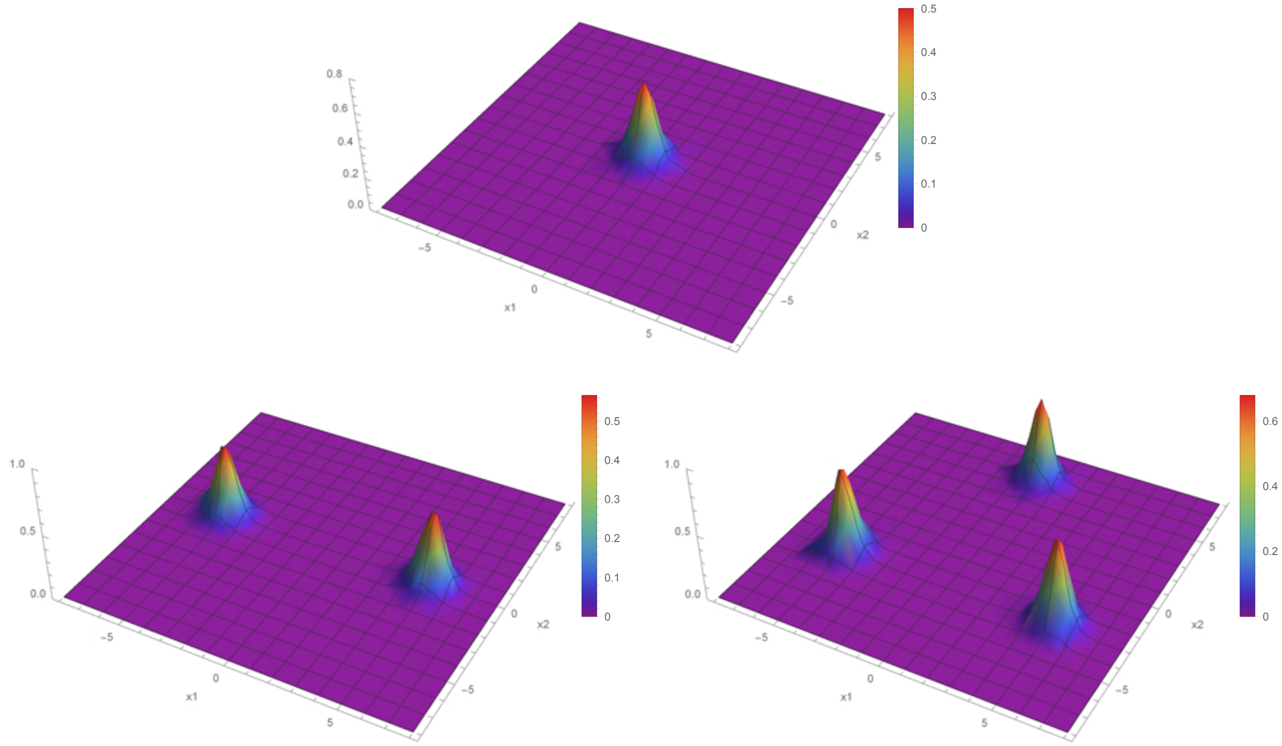


Figure 2.1: The charge density plots of the 't Hooft type solutions in eight dimensions ($n = 2$). The upper figure corresponds to $k = 1$, left and right ones in the lower figure correspond to $k = 2, 3$ respectively. All the plots are projected to a two-dimensional subspace in the eight-dimensional space.

In order to illustrate multi-instanton solutions, we write down the charge densities for $k = 1, 2, 3$ explicitly.

For $k = 1$, the charge density is calculated with using (2.72) (Of course, in this case,we can lead same result with using (2.52) also.):

$$Q^{(k=1)}_{\text{'t Hooft}} = -2^{2n-1}(4n)! \left(\frac{\lambda^2}{(\lambda^2 + ||\tilde{x}||^2)^2} \right)^{2n}. \tag{2.74}$$

This is same as the BPST one anti-instanton charge density, hence we find that $|Q^{(k=1)}_{\text{'t Hooft}}| = 1$. We note that the one-instanton solution that the 't Hooft ADHM data is singular at the instanton position:

$$A^{\text{singular}}_{\mu} = \frac{1}{4}\Sigma^{(+)}_{\mu\nu}\partial_\nu \ln\left(1 + \frac{\lambda^2}{||\tilde{x}||^2}\right), \tag{2.75}$$

while the BPST type solution (2.7) discussed in the previous section is non-singular. These solutions are connected by the following singular gauge transformation:

$$A^{\text{non-singular}}_{\mu} = g_1 A^{\text{singular}}_{\mu} g_1^{-1} + g_1\partial_\mu g_1^{-1}, \qquad g_1 = \frac{\tilde{x}}{\sqrt{||\tilde{x}||^2}}. \tag{2.76}$$

For $k = 2$ and $k = 3$, the approximate charge densities are evaluated as

$$Q^{(k=2)}_{\text{'t Hooft}} \simeq -(4n)!\cdot 2^{2n-1}\left(\frac{\lambda_1^2\|\tilde{x}_2\|^4+\lambda_2^2\|\tilde{x}_1\|^4+\lambda_1^2\lambda_2^2\left(\|\tilde{x}_1\|^2+\|\tilde{x}_2\|^2-2\tilde{x}_1^\mu\tilde{x}_2^\mu\right)}{\left(\lambda_1^2\|\tilde{x}_2\|^2+\lambda_2^2\|\tilde{x}_1\|^2+\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\right)^2}\right)^{2n}, \tag{2.77a}$$

$$\begin{aligned} Q^{(k=3)}_{\text{'t Hooft}} \simeq -(4n)!\cdot 2^{2n-1}\Big[\gamma\Big(&\lambda_1^2\|\tilde{x}_2\|^4\|\tilde{x}_3\|^4+\lambda_2^2\lambda_3^2\|\tilde{x}_1\|^4\left(\|\tilde{x}_2\|^2+\|\tilde{x}_3\|^2-2\tilde{x}_2^\mu\tilde{x}_3^\mu\right)\\ &+\lambda_2^2\|\tilde{x}_1\|^4\|\tilde{x}_3\|^4+\lambda_1^2\lambda_3^2\|\tilde{x}_2\|^4\left(\|\tilde{x}_1\|^2+\|\tilde{x}_3\|^2-2\tilde{x}_1^\mu\tilde{x}_3^\mu\right)\\ &+\lambda_3^2\|\tilde{x}_1\|^4\|\tilde{x}_2\|^4+\lambda_1^2\lambda_2^2\|\tilde{x}_3\|^4\left(\|\tilde{x}_1\|^2+\|\tilde{x}_2\|^2-2\tilde{x}_1^\mu\tilde{x}_2^\mu\right)\Big)\Big]^{2n}. \end{aligned} \tag{2.77b}$$

Here we have defined

$$\gamma=\frac{1}{\left(\lambda_1^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2+\lambda_2^2\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2+\lambda_3^2\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2+\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2\right)^2}. \tag{2.78}$$

For more detail on the calculations, please see below subsection 2.4.4.

The eight-dimenaional numerical profiles for the $k=1,2,3$ charge densities are found in Fig 2.1. Here the parameters that satisfy the well-separated limit (2.68) are chosen such that $a^\mu=0$, $\lambda=2$ for $k=1$, $a_1^1=-5$, $a_2^1=5$, $a_1^\mu=a_2^\mu=0$, $(\mu>1)$, $\lambda_1=\lambda_2=2$, for $k=2$ and $a_m^1=10/\sqrt{3}\times\sin(2\pi(m-1)/3)$, $a_m^2=10/\sqrt{3}\times\cos(2\pi(m-1)/3)$, $a_m^\mu=0$, $(\mu>2)$, $\lambda_m=2$, $(m=1,2,3)$ for $k=3$. For these parameters, the numerical results of instanton charges are evaluated as $Q\simeq 2\times 1.02$ $(k=2)$, $Q\simeq 3\times 1.03$ $(k=3)$. Therefore we find that the dilute instanton gas approximation, which is needed to solve the second ADHM constraint, works well.

Some comments are in order. First, we can find exact solutions to the condition (2.67), but these solutions are unsuitable data for multi-instantons. The condition is exactly solved by $\lambda_m=0$ (for all m), but this solution makes the pure gauge field, namely, it is a vacuum configuration. On the other hand, we find another exact solution $a_m=a_n$ $(m\neq n)$ which means that all the instantons are localized at the same point. However, this solution is equivalent to the one-instanton's one.

Second, we will show that the topological charge of the 't Hooft type k-instantons in the well-separated limit is an integer. In the case of $k\geq 2$, for the charge density formula (2.52), the charge density of the 't Hooft type in the well-separated limit is given by

$$Q_{\text{'t Hooft}}\simeq -2^{2n-1}(4n)!\sum_{p=1}^{k}\left(\frac{\lambda_p^2}{\left(\lambda_p^2+\|\tilde{x}_p\|^2\right)^2}\right)^{2n}. \tag{2.79}$$

This is the summation of the above mentioned one-instanton charge density, therefore we obtain $|Q_{\text{'t Hooft}}|=k$.

2.3.3 Jackiw-Nohl-Rebbi type ansatz

Let us study another of the 't Hooft solutions which is so-called Jackiw-Nohl-Rebbi (JNR) type solutions.

For (1.62), we give the JNR type ansatz as

$$\Delta=\begin{pmatrix}\mathbf{1}_{2^{2n-1}}\otimes\Lambda\\ \mathbf{1}_{2^{2n-1}}\otimes\mathbf{1}_k\end{pmatrix}\cdot x\otimes\mathbf{1}_k+\begin{pmatrix}-a_0\otimes\Lambda\\ \mathrm{diag}_{p=1}^k(-a_p)\end{pmatrix}=\begin{pmatrix}\tilde{x}_0\otimes\Lambda\\ \tilde{X}_{[2^{2n-1}k]\times[2^{2n-1}k]}\end{pmatrix}=e_\mu\otimes\begin{pmatrix}\tilde{x}_0^\mu\Lambda\\ \mathrm{diag}_{p=1}^k(\tilde{x}_p^\mu)\end{pmatrix}, \tag{2.80}$$

where $\Lambda=\begin{pmatrix}\lambda_1/\lambda_0 & \dots & \lambda_k/\lambda_0\end{pmatrix}$, $\tilde{x}_i=(x^\mu-a_i^\mu)e_\mu$, $a_i=a_i^\mu e_\mu$ and $\tilde{X}=\mathrm{diag}(\tilde{x}_1,\dots,\tilde{x}_k)$. Here $\lambda_i\in\mathbb{R}$ and $a_i^\mu\in\mathbb{R}$ $(i=0,\dots,k)$ are moduli parameters. We note that the JNR ansatz (2.80) is not in the canonical form and contain more moduli parameters than the 't Hooft one. The latter is obtained from the former by the limit $a_0\to\infty$, $\lambda_0\to\infty$ with fixed $a_0/\lambda_0=1$.

We can confirm that the JNR ansatz satisfies the first ADHM constraint (2.21):

$$\Delta^\dagger\Delta=\mathbf{1}_{2^{2n-1}}\otimes\left(\|\tilde{x}_0\|^2\,{}^t\Lambda\Lambda+\mathrm{diag}_{p=1}^k(\|\tilde{x}_p\|^2)\right)=\mathbf{1}_{2^{2n-1}}\otimes E_k^{(\mathrm{JNR})}, \tag{2.81}$$

where the symbol ${}^tM$ means the transposed matrix of M, so ${}^t\Lambda$ is k-column vector and ${}^t\Lambda\Lambda$ is $k\times k$ matrix. The solution to the Weyl equation (2.30) is given by

$$V=\frac{1}{\sqrt{\phi}}\begin{pmatrix}-\mathbf{1}_{2^{2n-1}}\\ \mathrm{diag}_{p=1}^k\left(\frac{\tilde{x}_p}{\|\tilde{x}_p\|^2}\right)\cdot\tilde{x}_0^\dagger\otimes{}^t\Lambda\end{pmatrix}, \tag{2.82}$$

where $\phi = 1 + \frac{\|\tilde{x}_0\|^2}{\lambda_0^2} \sum_{p=1}^k \left(\frac{\lambda_p^2}{\|\tilde{x}_p\|^2}\right)$.

Now we examine the second ADHM constraint. We first calculation $C^\dagger V,\ V^\dagger C$:

$$
\begin{aligned}
C^\dagger V &= \mathbf{1}_{2^{2n-1}} \otimes \begin{pmatrix} {}^t\Lambda & \mathbf{1}_k \end{pmatrix} \frac{1}{\sqrt{\phi}} \begin{pmatrix} -\mathbf{1}_{2^{2n-1}} \otimes 1 \\ e_\mu e_\nu^\dagger \otimes \left(\mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu}{\|\tilde{x}_p\|^2}\right) \tilde{x}_0^\nu \ {}^t\Lambda\right) \end{pmatrix} \\
&= \frac{1}{\sqrt{\phi}} \left(-\mathbf{1}_{2^{2n-1}} \otimes {}^t\Lambda + e_\mu e_\nu^\dagger \otimes \left(\mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu \tilde{x}_0^\nu}{\|\tilde{x}_p\|^2}\right) {}^t\Lambda\right)\right) \\
&= \frac{1}{\sqrt{\phi}} \left(e_\mu e_\nu^\dagger \otimes \left(-\delta^{\mu\#}\delta^{\nu\#}\mathbf{1}_k + \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu \tilde{x}_0^\nu}{\|\tilde{x}_p\|^2}\right)\right) {}^t\Lambda\right),
\end{aligned} \tag{2.83a}
$$

$$
V^\dagger C = \frac{1}{\sqrt{\phi}} \left(e_\nu e_\mu^\dagger \otimes \Lambda \left(-\delta^{\mu\#}\delta^{\nu\#}\mathbf{1}_k + \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu \tilde{x}_0^\nu}{\|\tilde{x}_p\|^2}\right)\right)\right). \tag{2.83b}
$$

Thus

$$
\begin{aligned}
C^\dagger V V^\dagger C &= \frac{1}{\phi} \left(e_\mu e_\nu^\dagger \otimes \left(-\delta^{\mu\#}\delta^{\nu\#}\mathbf{1}_k + \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu \tilde{x}_0^\nu}{\|\tilde{x}_p\|^2}\right)\right) {}^t\Lambda\right)\left(e_\rho e_\sigma^\dagger \otimes \Lambda\left(-\delta^{\rho\#}\delta^{\sigma\#}\mathbf{1}_k + \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\sigma \tilde{x}_0^\rho}{\|\tilde{x}_p\|^2}\right)\right)\right) \\
&= \frac{1}{\phi} \left(e_\mu e_\nu^\dagger e_\rho e_\sigma^\dagger \otimes \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\mu \tilde{x}_0^\nu}{\|\tilde{x}_p\|^2} - \delta^{\mu\#}\delta^{\nu\#}\right) \frac{1}{\lambda_0^2} \begin{pmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \dots & \lambda_1\lambda_k \\ \lambda_2\lambda_1 & \lambda_2^2 & \dots & \lambda_2\lambda_k \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 & \lambda_k\lambda_2 & \dots & \lambda_k^2 \end{pmatrix} \mathrm{diag}_{p=1}^k \left(\frac{\tilde{x}_p^\sigma \tilde{x}_0^\rho}{\|\tilde{x}_p\|^2} - \delta^{\sigma\#}\delta^{\rho\#}\right)\right) \\
&= \frac{1}{\phi\lambda_0^2} \left(e_\mu e_\nu^\dagger e_\rho e_\sigma^\dagger \otimes \begin{pmatrix} \lambda_1^2 Y_1^{\mu\nu} Y_1^{\sigma\rho} & \lambda_1\lambda_2 Y_1^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_1\lambda_k Y_1^{\mu\nu} Y_k^{\sigma\rho} \\ \lambda_2\lambda_1 Y_2^{\mu\nu} Y_1^{\sigma\rho} & \lambda_2^2 Y_2^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_2\lambda_k Y_2^{\mu\nu} Y_k^{\sigma\rho} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 Y_k^{\mu\nu} Y_1^{\sigma\rho} & \lambda_k\lambda_2 Y_k^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_k^2 Y_k^{\mu\nu} Y_k^{\sigma\rho} \end{pmatrix}\right) \\
&=: \frac{1}{\phi\lambda_0^2} \left(e_\mu e_\nu^\dagger e_\rho e_\sigma^\dagger \otimes E_{(\mathrm{JNR})}^{\mu\nu\rho\sigma}\right),
\end{aligned} \tag{2.84}
$$

where $\# := 4n$ and

$$
E_{(\mathrm{JNR})}^{\mu\nu\rho\sigma} = \begin{pmatrix} \lambda_1^2 Y_1^{\mu\nu} Y_1^{\sigma\rho} & \lambda_1\lambda_2 Y_1^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_1\lambda_k Y_1^{\mu\nu} Y_k^{\sigma\rho} \\ \lambda_2\lambda_1 Y_2^{\mu\nu} Y_1^{\sigma\rho} & \lambda_2^2 Y_2^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_2\lambda_k Y_2^{\mu\nu} Y_k^{\sigma\rho} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_k\lambda_1 Y_k^{\mu\nu} Y_1^{\sigma\rho} & \lambda_k\lambda_2 Y_k^{\mu\nu} Y_2^{\sigma\rho} & \dots & \lambda_k^2 Y_k^{\mu\nu} Y_k^{\sigma\rho} \end{pmatrix}, \tag{2.85}
$$

Here we have defined $Y_m^{\mu\nu} = \tilde{x}_m^\mu \tilde{x}_0^\nu / \|\tilde{x}_m\|^2 - \delta^{\mu\#}\delta^{\nu\#}$ and $m = 1, \dots, k$ is not summed. In each component in the matrix in (2.85), we have

$$
Y_m^{\mu\nu} Y_n^{\sigma\rho} e_\mu e_\nu^\dagger e_\rho e_\sigma^\dagger = \frac{\|\tilde{x}_0\|^2}{\|\tilde{x}_m\|^2 \|\tilde{x}_n\|^2} \tilde{x}_m \tilde{x}_n^\dagger - \frac{1}{\|\tilde{x}_n\|^2} \tilde{x}_0 \tilde{x}_n^\dagger - \frac{1}{\|\tilde{x}_m\|^2} \tilde{x}_m \tilde{x}_0^\dagger + \mathbf{1}_{2^{2n-1}}. \tag{2.86}
$$

For $k = 1$, since we have the relation $\tilde{x}_a \tilde{x}_b^\dagger + \tilde{x}_b \tilde{x}_a^\dagger = 2\tilde{x}_a^\mu \tilde{x}_b^\mu \mathbf{1}_{2^{2n-1}}$, the right-hand side of (2.86) is proportional to $\mathbf{1}_{2^{2n-1}}$ and the second ADHM constraint is satisfied. The charge density of the $k = 1$ JNR solution is given by

$$
Q_{\mathrm{JNR}}^{(k=1)} = -(4n)! \cdot 2^{2n-1} \left(\frac{\overline{\lambda}_1^2 \left(\|\tilde{x}_1\|^2 + \|\tilde{x}_0\|^2 - 2\tilde{x}_0^\mu \tilde{x}_1^\mu\right)}{\left(\|\tilde{x}_0\|^2 \overline{\lambda}_1^2 + \|\tilde{x}_1\|^2\right)^2}\right)^{2n}, \tag{2.87}
$$

where $\overline{\lambda}_m = \lambda_m/\lambda_0$. The moduli parameters are $\lambda_1/\lambda_0 = \lambda$ and $a_1^\mu - a_0^\mu = a^\mu$, so the $k = 1$ JNR solution has total $(4n + 1)$ parameters. Therefore the $k = 1$ JNR data is essentially equal to the $k = 1$ 't Hooft data, and we find that the numerical results of the $k = 1$ instanton charge (2.87) is $Q = 1$.

For $k \geq 2$ case, it is not straightforward to solve the constraint (2.53) in a general fashion. However, a solution is found in the well-separated limit (2.68). In this limit, we can neglect all the off-diagonal components in $E_k^{(\mathrm{JNR})}$:

$$
E_k^{(\mathrm{JNR})} \simeq \mathrm{diag}_{p=1}^k \left(\|\tilde{x}_0\|^2 \overline{\lambda}_p^2 + \|\tilde{x}_p\|^2\right). \tag{2.88}
$$

Then, the second ADHM constraint is satisfied:

$$C^\dagger \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C \simeq \mathbf{1}_{2^{2n-1}} \otimes \mathrm{diag}^k_{p=1}\left(\frac{\|\tilde{x}_0\|^2\overline{\lambda}_p^4 + 2\overline{\lambda}_p^2\tilde{x}_0^\mu\tilde{x}_p^\mu + \|\tilde{x}_p\|^2}{\|\tilde{x}_0\|^2\overline{\lambda}_p^2 + \|\tilde{x}_p\|^2}\right). \tag{2.89}$$

We also observe that the instanton charge is quantized in this limit by using the same formula of the 't Hooft ones. We note that the JNR data is not in the canonical form. In this case, the charge density formula (2.52) is rewritten as

$$Q = -(4n)! \cdot 2^{2n-1}\mathrm{Tr}_k\left(\left(E_k^{(1)}\right)^{-1}(C^{(2)} - E_k^{(2)})\right)^{2m}, \tag{2.90}$$

where $C^{(2)}$ is defined by $C^\dagger C = \mathbf{1}_{2^{2n-1}} \otimes C^{(2)}$. In the well-separated limit (2.68), we have $C^\dagger C \simeq \mathbf{1}_{2^{2n01}} \otimes (\mathrm{diag}^k_{p=1}\overline{\lambda}_p^2 + \mathbf{1}_k)$ thus

$$Q_{\mathrm{JNR}} \simeq -(4n)! \cdot 2^{2n-1}\sum_{p=1}^{k}\left(\frac{\overline{\lambda}_p^2\left(\|\tilde{x}_p\|^2 + \|\tilde{x}_0\|^2 - 2\tilde{x}_0^\mu\tilde{x}_p^\mu\right)}{\left(\|\tilde{x}_0\|^2\overline{\lambda}_p^2 + \|\tilde{x}_p\|^2\right)^2}\right)^{2n}. \tag{2.91}$$

This is just the summation of the JNR type one-instanton charge density and the charge associated with (2.91) is $Q = k$.

Note that we previously discussed these ADHM data with the unitary gauge group for simplify, we can take the orthogonal gauge group case as following. Recall the fact that the gauge transformation of the ADHM instantons is derived from the degree of freedom of the zero mode, thus the gauge group G depend the (field) values of the zero mode (ref.(2.46) and (2.47)). In previous deiscussion, we assumed that the zero mode takes complex values because the ASD basis e_μ was generated by the complex Clifford algebra $C\ell_{4n-1}(\mathbb{C})$ namely that the Weyl operater take complex values. Strictly speaking, this fact comes from that the explicit matrix representations of the complex Clifford algebra $C\ell_{4n-1}(\mathbb{C})$ are the comple values matrices (see subsection 2.5). However we can take the real Clifford algebra $C\ell_{4n-1}(\mathbb{R})$ instead of the complex Clifford algebra $C\ell_{4n-1}(\mathbb{C})$ to construct the ASD basis (see detail discussion in subsection 2.5). In this case, the explicit matrix representations of the real Clifford algebra $C\ell_{4n-1}(\mathbb{R})$ are the real values matrices namely the ASD basis take real values. If ADHM data take real values then the Weyl operator is the real values matrices, thus the zero mode take real values namely the gauge group is the orthogonal group. Now we recall that these three type ADHM data (BPST type, 't Hooft type and JNR type) take real values, and these data (with well-separated condition) satisfy the ADHM constraints [3]. Therefore we can choose the gauge group G by using the Clifford algebra types.

2.4 The detailed calculations

2.4.1 The gauge field for anti-Hermite versus Hermite

Here we discuss the difference between the anti-Hermite gauge ($A_\mu^\dagger = -A_\mu$) and the Hermite gauge ($A_\mu^\dagger = A_\mu$). The anti-Hermite gauge leads the equation that is more simple form than the Hermite gauge. Hence the anti-Hermite gauge is usually used in the mathematical physics. On the other hand, the Hermite gauge leads pysical quantities in which real values (Of course, the anti-Hermite gauge bacomes pure imaginaly physical quantities).

The field strength is defined by

$$\text{anti-Hermite}: F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + \left[A_\mu, A_\nu\right] \;\Rightarrow\; F^\dagger_{\mu\nu} = -F_{\mu\nu}, \tag{2.92a}$$

$$\text{Hermite}: F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i\left[A_\mu, A_\nu\right] \;\Rightarrow\; F^\dagger_{\mu\nu} = F_{\mu\nu}, \tag{2.92b}$$

We naturally suppose that the action 2.4 becomes minumum when gauge field satisfies the ASD equation (2.1). Hence we decide the factor of the action by demanding the condition that the action increase positive if the gauge field out of the ASD equation. Now we recall the Bogomol'nyi completion:

$$S = \int \mathrm{Tr}F(n) \wedge *_{4n}F(n) = \frac{1}{2}\int \mathrm{Tr}\left(F(n) \pm *_{4n}F(n)\right)^2 \mp 2F(2n) \geq \int \mathrm{Tr}F(2n). \tag{2.93}$$

[3] Here the reason that we expressly wrote "the ADHM constraint" stress that we did not use the explicit form of the ASD basis when we confirm the validity of the ADHM data. The explicit matrix of ASD basis is deferent between the real and complex Clifford algebra, thus if we use the explicit form when confirm the validity of the ADHM data then we do not hold the guarantee of data when using another Clifford algebra in usually. Indeed, the ADHM equations are deferent form between the real and complex Clifford algebra.

For this equation, the action S increase positive if and only if $\mathrm{Tr}\,(F(n) \pm *_{4n}F(n))^2 \geq 0$. Thus we determine the action factor by the condition that the arbitrary field strength F satisfies $\mathrm{Tr}\,(F(n) \pm *_{4n}F(n))^2 \geq 0$. First, $F(n)^\dagger$ becomes

$$\text{anti-Hermite} : F(n)^\dagger = F^\dagger \wedge \cdots \wedge F^\dagger = -F \wedge \cdots \wedge -F = (-1)^n F(n), \tag{2.94a}$$

$$\text{Hermite} : F(n)^\dagger = F^\dagger \wedge \cdots \wedge F^\dagger = F \wedge \cdots \wedge F = F(n), \tag{2.94b}$$

and this Hodge dual becomes

$$\text{anti-Hermite} : *_{4n}F(n)^\dagger = (-1)^n *_{4n} F(n). \tag{2.95a}$$

$$\text{Hermite} : *_{4n}F(n)^\dagger = *_{4n}F(n). \tag{2.95b}$$

Thus

$$\text{anti-Hermite} : (F(n) + *_{4n}F(n))^\dagger = F(n)^\dagger + *_{4n}F(n)^\dagger = (-1)^n \left(F(n) + *_{4n}F(n)\right), \tag{2.96a}$$

$$\text{Hermite} : (F(n) + *_{4n}F(n))^\dagger = F(n)^\dagger + *_{4n}F(n)^\dagger = F(n) + *_{4n}F(n). \tag{2.96b}$$

Now we recall the following

$$X \text{ is anti-Hermite} \Rightarrow \mathrm{Tr}X^2 \leq 0 \in \mathbb{R}, \tag{2.97a}$$

$$X \text{ is Hermite} \Rightarrow \mathrm{Tr}X^2 \geq 0 \in \mathbb{R}, \tag{2.97b}$$

For (2.96) and (2.97), the factor of the action is defined by

$$A_\mu : \text{anti-Hermite} \Rightarrow \begin{cases} (F(n) + *_{4n}F(n))^2 \leq 0 & \text{at } n \in \mathbb{O}, \ \Rightarrow S = -\int \mathrm{Tr}F(n) \wedge *_{4n}F(n), \\ (F(n) + *_{4n}F(n))^2 \geq 0 & \text{at } n \in \mathbb{E}. \ \Rightarrow S = \int \mathrm{Tr}F(n) \wedge *_{4n}F(n), \end{cases}$$

$$\Longleftrightarrow \ S = (-1)^n \int \mathrm{Tr}F(n) \wedge *_{4n}F(n), \tag{2.98a}$$

$$A_\mu : \text{Hermite} \Rightarrow (F(n) + *_{4n}F(n))^2 \geq 0 \Rightarrow S = \int \mathrm{Tr}F(n) \wedge *_{4n}F(n). \tag{2.98b}$$

Next we consider the ADHM construction case.

$$\text{anti-Hermite} : A_\mu = V^\dagger \partial_\mu V, \tag{2.99a}$$

$$\text{Hermite} : A_\mu = iV^\dagger \partial_\mu V. \tag{2.99b}$$

In the Hermite gauge case, the calculation of the field strength from (2.99) becomes

$$\begin{aligned} F_\mu &= \partial_\mu A_\nu - iA_\mu A_\nu - (\mu \leftrightarrow \nu) \\ &= i\partial_\mu V^\dagger \partial_\nu V + i\partial_\mu V^\dagger V V^\dagger \partial_\nu V - (\mu \leftrightarrow \nu) \qquad \because).V^\dagger \partial_\nu V = -\partial_\mu V^\dagger V \\ &= i\left(\partial_\mu V^\dagger \left(1 - VV^\dagger\right) \partial_\nu V\right) - (\mu \leftrightarrow \nu) \\ &= \dots \\ &= V^\dagger C \left(\Delta^\dagger \Delta\right)^{-1} \left(i\Sigma^{(-)}_{\mu\nu} \otimes \mathbf{1}_k\right) C^\dagger V. \end{aligned} \tag{2.100}$$

Hence the charge density: Q = components $\mathrm{Tr}F(2n)$ becomes

$$Q_{\text{anti-Hermite}} = i^{2n} Q_{\text{Hermite}} = (-1)^n Q_{\text{Hermite}}. \tag{2.101}$$

Therefore the charge density formulas are

$$\text{anti-Hermite} : Q = \pm(-1)^n (4n)! \mathrm{Tr}_N \left(V^\dagger C \left(\Delta^\dagger \Delta\right)^{-1} C^\dagger V\right)^{2n}, \tag{2.102a}$$

$$\text{Hermite} : Q = \pm(4n)! \mathrm{Tr}_N \left(V^\dagger C \left(\Delta^\dagger \Delta\right)^{-1} C^\dagger V\right)^{2n}, \tag{2.102b}$$

and

$$\text{anti-Hermite} : Q = \pm(-1)^n (4n)! 2^{2n-1} \mathrm{Tr}_k \left(E_k^{-1} \left(\mathbf{1}_k - E_k^{(2)}\right)\right), \tag{2.103a}$$

$$\text{Hermite} : Q = \pm(4n)! 2^{2n-1} \mathrm{Tr}_k \left(E_k^{-1} \left(\mathbf{1}_k - E_k^{(2)}\right)\right). \tag{2.103b}$$

2.4.2 ASD tensor

In this subsection, we will show the (anti-)self-duality of ASD tensor (2.12) and the whole product of ASD tenosr (2.13).

Note that we here do not use the explicit form of the Clifford algebra, therefore the following relations are general properties of the ASD basis.

For latter convenience, we introduce the following:

$$\gamma_i := \Gamma_i^{(+)} = -\Gamma_i^{(-)} \quad \text{where } \Gamma_i^{(\pm)} = P_\pm \Gamma_i, \tag{2.104}$$

Thus

$$e_\mu = \delta_{\mu 4n} 1 - \delta_{\mu j}\gamma_j, \quad e_\mu^\dagger = \delta_{\mu 4n} 1 + \delta_{\mu j}\gamma_j, \tag{2.105}$$

Proposition 2.4.1 (The basic property of the ASD basis e_μ**).**

$$e_\mu e_\nu^\dagger + e_\nu e_\mu^\dagger = e_\mu^\dagger e_\nu + e_\nu^\dagger e_\mu = 2\delta_{\mu\nu}, \tag{2.106a}$$

$$e_\mu e_\nu + e_\nu e_\mu = 2\delta_{\mu 4n} e_\nu + 2\delta_{\nu 4n} e_\mu - 2\delta_{\mu\nu}, \tag{2.106b}$$

$$e_\mu^\dagger e_\nu^\dagger + e_\nu^\dagger e_\mu^\dagger = 2\delta_{\mu 4n} e_\nu^\dagger + 2\delta_{\nu 4n} e_\mu^\dagger - 2\delta_{\mu\nu}. \tag{2.106c}$$

Proof. The proof of (2.106a). Here we consider the case of $e_\mu e_\nu^\dagger + e_\nu e_\mu^\dagger$. For (2.105),

$$\begin{aligned} e_\mu e_\nu^\dagger + e_\nu e_\mu^\dagger &= (\delta_{\mu 4n} - \delta_{\mu i}\gamma_i)(\delta_{\nu 4n} + \delta_{\nu j}\gamma_j) + (\delta_{\nu 4n} - \delta_{\nu i}\gamma_i)(\delta_{\mu 4n} + \delta_{\mu j}\gamma_j) \\ &= 2\delta_{\mu 4n}\delta_{\nu 4n} - \delta_{\mu i}\delta_{\nu j}\gamma_i\gamma_j - \delta_{\nu i}\delta_{\mu j}\gamma_i\gamma_j \\ &= 2\delta_{\mu 4n}\delta_{\nu 4n} - \delta_{\mu i}\delta_{\nu j}\underbrace{\left(\gamma_i\gamma_j + \gamma_j\gamma_i\right)}_{-2\delta_{ij}} \\ &= 2\delta_{\mu 4n}\delta_{\nu 4n} + 2\delta_{\mu i}\delta_{\nu j}\delta_{ij} = 2\delta_{\mu\nu}. \end{aligned} \tag{2.107}$$

Note that the proof of $e_\mu^\dagger e_\nu + e_\nu^\dagger e_\mu$ is same method ($e_\mu^\dagger e_\nu + e_\nu^\dagger e_\mu = (\delta_{\mu 4n} + \delta_{\mu i}\gamma_i)(\delta_{\nu 4n} - \delta_{\nu j}\gamma_j) + (\delta_{\nu 4n} + \delta_{\nu i}\gamma_i)(\delta_{\mu 4n} - \delta_{\mu j}\gamma_j) = \dots$).

The proof of (2.106b). For (2.105),

$$\begin{aligned} e_\mu e_\nu + e_\nu e_\mu &= (\delta_{\mu 4n} - \delta_{\mu i}\gamma_i)(\delta_{\nu 4n} - \delta_{\nu j}\gamma_j) + (\delta_{\nu 4n} - \delta_{\nu i}\gamma_i)(\delta_{\mu 4n} - \delta_{\mu j}\gamma_h) \\ &= 2\delta_{\mu 4n}\delta_{\nu 4n} - 2\delta_{\mu 4n}\delta_{\nu i}\gamma_i - 2\delta_{\nu 4n}\delta_{\mu i}\gamma_i + \delta_{\mu i}\delta_{\nu j}\left(\gamma_i\gamma_j + \gamma_j\gamma_i\right) \\ &= 2\delta_{\mu 4n} e_\nu + 2\delta_{\nu 4n} e_\mu - 2\delta_{\mu\nu}. \end{aligned} \tag{2.108}$$

(2.106c) is able to prove with same method. *Q.E.D.*

Using (2.106a), we obtatin the relation with $\mu \neq \nu$:

$$\Sigma_{\mu\nu}^{(+)} = 2e_\mu^\dagger e_\nu = 2\Gamma_\mu^{(+)}\Gamma_\nu^{(-)}, \quad \Sigma_{\mu\nu}^{(-)} = 2e_\mu e_\nu^\dagger = 2\Gamma_\mu^{(-)}\Gamma_\nu^{(+)}, \qquad \mu \neq \nu \tag{2.109}$$

where we introduced $\Gamma_{4n}^{(\pm)} = 1$. Since $\Gamma_{4n}^{(\pm)}\Gamma_i^{(\mp)} = \Gamma_i^{(\mp)} = -\Gamma_i^{(\pm)} = -\Gamma_i^{(\pm)}\Gamma_{4n}^{(\mp)}$, thus we obtain

$$\Gamma_\mu^{(\pm)}\Gamma_\nu^{(\mp)} = -\Gamma_\nu^{(\pm)}\Gamma_\mu^{(\mp)}. \tag{2.110}$$

Now we give a lemma which is used to lead the (anti-)self-duality of ASD tensor (2.12).

Lemma 2.4.1.
Here $\Sigma_{\mu\nu}^{(\pm)}$ is the ASD tensor which is defined by (2.8)*, then ASD tensor satisfies the following relation:*

$$\Sigma_{[\mu_1\mu_2}^{(\pm)} \dots \Sigma_{\mu_{2n-1}\mu_{2n}]}^{(\pm)} = \Sigma_{\mu_1\mu_2}^{(\pm)} \dots \Sigma_{\mu_{2n-1}\mu_{2n}}^{(\pm)}, \tag{2.111}$$

where $\mu_i \neq \mu_j$

Proof. We use the mathematical induction.

Let the number m denote the number of $\Gamma_i^{(\pm)}$ product times (i.e. $2m = n$). First case of $m = 2$, from (2.109) and (2.110),

$$\Sigma^{(\pm)}_{[\mu\nu]} = 2\Gamma^{(\pm)}_{[\mu}\Gamma^{(\mp)}_{\nu]} = \Gamma^{(\pm)}_{\mu}\Gamma^{(\mp)}_{\nu} - \Gamma^{(\pm)}_{\nu}\Gamma^{(\mp)}_{\mu} = \Gamma^{(\pm)}_{\mu}\Gamma^{(\mp)}_{\nu} + \Gamma^{(\pm)}_{\mu}\Gamma^{(\mp)}_{\nu} = 2\Gamma^{(\pm)}_{\mu}\Gamma^{(\mp)}_{\nu} = \Sigma^{(\pm)}_{\mu\nu}. \tag{2.112}$$

If the case of $m = p - 1$ is satisfies

$$p - 1 \in \text{even} : \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]} = \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}}, \tag{2.113a}$$

$$p - 1 \in \text{odd} : \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]} = \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}}, \tag{2.113b}$$

then the case of $m = p$ becomes

$$\begin{aligned}
p - 1 \in \text{even} : \quad & \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}}\Gamma^{(\pm)}_{\mu_p]} = \frac{1}{p}\left(\Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]}\Gamma^{(\pm)}_{\mu_p} - \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\widehat{\mu_p}}\Gamma^{(\pm)}_{\mu_{p-1}]} + \dots + \Gamma^{(\pm)}_{\mu_p}\Gamma^{(\mp)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]}\right) \\
&= \frac{1}{p}\left(\Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]}\Gamma^{(\pm)}_{\mu_p} + \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]}\Gamma^{(\pm)}_{\mu_p} + \dots + \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]}\Gamma^{(\pm)}_{\mu_p}\right) \\
&= \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]}\Gamma^{(\pm)}_{\mu_p}, \\
&\therefore (2.113\text{a}) \Rightarrow \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}}\Gamma^{(\pm)}_{\mu_p]} = \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}}\Gamma^{(\pm)}_{\mu_p}, \qquad (2.114\text{a}) \\
p - 1 \in \text{odd} : \quad & \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}}\Gamma^{(\mp)}_{\mu_p]} = \frac{1}{p}\left(\Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]}\Gamma^{(\mp)}_{\mu_p} - \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\widehat{\mu_p}}\Gamma^{(\mp)}_{\mu_{p-1}]} + \dots + \Gamma^{(\pm)}_{\mu_p}\Gamma^{(\mp)}_{[\mu_1} \dots \Gamma^{(\mp)}_{\mu_{p-1}]}\right) \\
&= \frac{1}{p}\left(\Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]}\Gamma^{(\mp)}_{\mu_p} + \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]}\Gamma^{(\mp)}_{\mu_p} + \dots + \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]}\Gamma^{(\mp)}_{\mu_p}\right) \\
&= \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}]}\Gamma^{(\mp)}_{\mu_p}, \\
&\therefore (2.113\text{b}) \Rightarrow \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}}\Gamma^{(\mp)}_{\mu_p]} = \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\pm)}_{\mu_{p-1}}\Gamma^{(\mp)}_{\mu_p}, \qquad (2.114\text{b})
\end{aligned}$$

Now the case of $m = 2$ satisfies (2.113a), thus the all cases of $m \in \mathbb{N}$ satisfy (2.114a) or (2.114b).

$$\begin{aligned}
\Sigma^{(\pm)}_{[\mu_1\mu_2} \dots \Sigma^{(\pm)}_{\mu_{2n-1}\mu_{2n}]} = \Sigma^{(\pm)}_{\mu_1\mu_2} \dots \Sigma^{(\pm)}_{\mu_{2n-1}\mu_{2n}} \iff & \left(\frac{1}{2}\right)^n \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{2n-1}}\Gamma^{(\mp)}_{\mu_{2n}]} = \left(\frac{1}{2}\right)^n \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\pm)}_{\mu_{2n-1}}\Gamma^{(\mp)}_{\mu_{2n}} \\
\iff & \Gamma^{(\pm)}_{[\mu_1} \dots \Gamma^{(\pm)}_{\mu_{2n-1}}\Gamma^{(\mp)}_{\mu_{2n}]} = \Gamma^{(\pm)}_{\mu_1} \dots \Gamma^{(\pm)}_{\mu_{2n-1}}\Gamma^{(\mp)}_{\mu_{2n}}.
\end{aligned} \tag{2.115}$$

Therefore the all cases of $n \in \mathbb{N}$ satisfy (2.111) because (2.114b). $\mathcal{Q.E.D.}$

Next we give the Hodge duality of the ASD tensor in $4n$ dimensions. This duality is the generalization of the (anti-)self-duality, and the (anti-)self-duality (2.12) follows from the following theorem.

Theorem 2.4.1 (The Hodge duality of the $4n$-dimensional ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$)**.**
Here $\Sigma^{(\pm)}_{\mu\nu}$ is the ASD tensor which is defined by (2.8)*, then ASD tensor satisfies the* $4n$*-dimensional Hodge duality relation:*

$$\Sigma^{(\pm)}_{[\mu_1\mu_2} \dots \Sigma^{(\pm)}_{\mu_{2p-1}\mu_{2p}]} = \pm\frac{(-1)^{n-p}}{2^{2(n-p)}(2(2n-p))!}\varepsilon_{\mu_1\mu_2\dots\mu_{2p}\mu_{2p+1}\dots\mu_{4n-1}\mu_{4n}}\Sigma^{(\pm)}_{\mu_{2p+1}\mu_{2p+2}} \dots \Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} \tag{2.116}$$

where p is integer as $0 < p < 2n$.

Proof. Now we write the matrix reparesentation of $\Gamma_i = \Gamma^{(+)}_i \oplus \Gamma^{(-)}_i$ as

$$\Gamma_i = \begin{pmatrix} \Gamma^{(+)}_i & 0 \\ 0 & \Gamma^{(-)}_i \end{pmatrix}. \tag{2.117}$$

The volume(chirality) element ω which is defined by (2.10) is the central element of the Clifford algebra $C\ell_{4n-1}(K)$:

$$\Gamma_1\Gamma_2\dots\Gamma_{4n-1} = (-1)^{n+1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.118}$$

The proof of the Hodge duality start from this equation. For the following proof, we will introduce the notation that an basic order and a lexicographic order. The basic order means that the numbers on index always increase toward the right side, for example $X_{12\dots56}$. The lexicographic order means that the index numbers increase toward the right side, for example $X_{\mu_1\mu_2\dots\mu_5\mu_6}$ (In this order, index is not number).

The strategy of the proof is that we first prove the basic order case which is more easily than the lexicographic order case. Next we prove the lexicographic order case which can be proved with using similar method of the above case.

First, we right multiply $\Gamma_{4n-1}\to\Gamma_{4n-2}\to\dots\to\Gamma_{2p+1}$ in order on both sides (2.118) and rewrite the r.h.s. to the basic order.

$$\begin{aligned}(-1)\Gamma_1\Gamma_2\dots\Gamma_{2p} &= (-1)^{n+1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Gamma_{4n-1}\Gamma_{4n-2}\dots\Gamma_{2p+1},\\ &= (-1)^{n+1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\times(-1)^{-p-1}\Gamma_{2p+1}\Gamma_{2p+2}\dots\Gamma_{4n-1}.\end{aligned} \tag{2.119}$$

Here the origin of the l.h.s. (overall) factor (-1) is $(2(2n-p)-1)$ times product of $\Gamma_i^2=-1$ ($i=2p+1\dots,4n-1$ and not summed). We give the explain of the r.h.s. factor $(-1)^{p+1}$ as follows. For $\Gamma_i\Gamma_j=-\Gamma_j\Gamma_i$ ($i\neq j$), we taken the operate that $\Gamma_{4n-1}\Gamma_{4n-2}\dots\Gamma_{2p+1}\to(-1)^{2(2n-p)-2}\Gamma_{4n-2}\dots\Gamma_{2p+1}\Gamma_{4n-1}\to(-1)^{2(2n-p)-2}(-1)^{2(2n-p)-3}\Gamma_{4n-3}\dots\Gamma_{2n+1}\Gamma_{4n-2}\Gamma_{4n-1}\to\dots$, thus the exponent part of (-1) is summation $(-2)+(-3)+\dots+(-2(2n-p))$ (Here we omit the common even number $2(2n-p)$ because of $(-1)^{2(2n-p)}=1$ for all p.). Because of $(-2)+(-3)+\dots+(-2(2n-p))=-\sum_{i=2}^{2(2n-p)}i=-(2(2n-p)^2+(2n-p)-1)$ and $(-1)^{-m}=(-1)^m$ $m\in\mathbb{N}$, the factor becomes $(-1)^{-(2(2n-p)^2+(2n-p)-1)}=(-1)^{-p-1}$.

Now we using (2.117),

$$\begin{aligned}&(-1)\Gamma_1\Gamma_2\dots\Gamma_{2p} = (-1)^{n+1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\times(-1)^{-p-1}\Gamma_{2p+1}\Gamma_{2p+2}\dots\Gamma_{4n-1}\\ \Longleftrightarrow&\begin{pmatrix}\Gamma_1^{(+)}\Gamma_2^{(+)}\dots\Gamma_{2p}^{(+)} & \\ 0 & \Gamma_1^{(-)}\Gamma_2^{(-)}\dots\Gamma_{2p}^{(-)}\end{pmatrix} = (-1)^{n-p+1}\begin{pmatrix}\Gamma_{2p+1}^{(+)}\Gamma_{2p+2}^{(+)}\dots\Gamma_{4n-1}^{(+)} & 0\\ 0 & -\Gamma_{2p+1}^{(-)}\Gamma_{2p+2}^{(-)}\dots\Gamma_{4n-1}^{(-)}\end{pmatrix}\end{aligned} \tag{2.120}$$

We right multiply $\Gamma_{4n}=1$ on the r.h.s., and then rewrite $\Gamma_\mu^{(\pm)}$ to the ASD tensor $\Sigma_{\mu\nu}^{(\pm)}$ with using (2.109) and $\Gamma_i^{(+)}=-\Gamma_i^{(-)}$:

$$\begin{aligned}&\begin{pmatrix}\Gamma_1^{(+)}\Gamma_2^{(+)}\dots\Gamma_{2p}^{(+)} & \\ 0 & \Gamma_1^{(-)}\Gamma_2^{(-)}\dots\Gamma_{2p}^{(-)}\end{pmatrix} = (-1)^{n-p+1}\begin{pmatrix}\Gamma_{2p+1}^{(+)}\Gamma_{2p+2}^{(+)}\dots\Gamma_{4n}^{(+)} & 0\\ 0 & -\Gamma_{2p+1}^{(-)}\Gamma_{2p+2}^{(-)}\dots\Gamma_{4n}^{(-)}\end{pmatrix}\\ \Longleftrightarrow&\begin{cases}(-1)^p2^{-p}\Sigma_{12}^{(+)}\dots\Sigma_{(2p-1)(2p)}^{(+)} = (-1)^{n-p+1}\times(-1)^{2n-p-1}2^{-(2n-p)}\Sigma_{(2p+1)(2p+2)}^{(+)}\dots\Sigma_{(4n-1)(4n)}^{(+)}\\ (-1)^p2^{-p}\Sigma_{12}^{(-)}\dots\Sigma_{(2p-1)(2p)}^{(-)} = (-1)^{n-p+1}\times-1\times(-1)^{2n-p-1}2^{-(2n-p)}\Sigma_{(2p+1)(2p+2)}^{(-)}\dots\Sigma_{(4n-1)(4n)}^{(-)}\end{cases}\\ \Longleftrightarrow&\ \Sigma_{12}^{(\pm)}\dots\Sigma_{(2p-1)(2p)}^{(\pm)} = \pm(-1)^{n-p}2^{-2(n-p)}\Sigma_{(2p+1)(2p+2)}^{(\pm)}\dots\Sigma_{(4n-1)(4n)}^{(\pm)}.\end{aligned} \tag{2.121}$$

Since the lemma 2.4.1, we obtain

$$\Sigma_{[12}^{(\pm)}\dots\Sigma_{(2p-1)(2p)]}^{(\pm)} = \pm(-1)^{n-p}2^{-2(n-p)}\Sigma_{(2n+1)(2n+2)}^{(\pm)}\dots\Sigma_{(4n-1)(4n)}^{(\pm)}. \tag{2.122}$$

Next we consider the general case. For $\left\{\Gamma_i,\Gamma_j\right\}=-2\delta_{ij}$,

$$\Gamma_{\mu_1}\Gamma_{\mu_2}\dots\Gamma_{\mu_{4n-1}} = \varepsilon_{\mu_1\mu_2\dots\mu_{4n-1}}\Gamma_1\Gamma_2\dots\Gamma_{4n-1}, \tag{2.123}$$

thus

$$\Gamma_{\mu_1}\Gamma_{\mu_2}\dots\Gamma_{\mu_{4n-1}} = (-1)^{n+1}\varepsilon_{\mu_1\mu_2\dots\mu_{4n-1}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.124}$$

We start from this equation. We right multiply $\Gamma_{\mu_{4n-1}}\Gamma_{\mu_{4n-2}}\dots\Gamma_{\mu_{2p+1}}$ in both sides, and then we rewrite the r.h.s. in lexicographic order.

$$\begin{aligned}(-1)\Gamma_{\mu_1}\Gamma_{\mu_2}\dots\Gamma_{\mu_{2p}} &= (-1)^{n+1}\varepsilon_{\mu_1\mu_2\dots\mu_{2p}\mu_{2p+1}\dots\mu_{4n-1}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\Gamma_{\mu_{4n-1}}\Gamma_{\mu_{4n-2}}\dots\Gamma_{\mu_{2p+1}}\\ &= (-1)^{n+1}\varepsilon_{\mu_1\mu_2\dots\mu_{2p}\mu_{2p+1}\dots\mu_{4n-1}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\times(-1)^{-p-1}\Gamma_{\mu_{2p+1}}\Gamma_{\mu_{2p+2}}\dots\Gamma_{\mu_{4n-1}},\end{aligned} \tag{2.125}$$

where μ_i is not summed. Now we using (2.117),

$$(-1)\Gamma_{\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_{2p}} = (-1)^{n-p}\varepsilon_{\mu_1\mu_2\ldots\mu_{2p}\mu_{2p+1}\ldots\mu_{4n-1}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\times\Gamma_{\mu_{2p+1}}\Gamma_{\mu_{2p+2}}\ldots\Gamma_{\mu_{4n-1}}$$
$$\iff \begin{pmatrix}\Gamma^{(+)}_{\mu_1}\Gamma^{(+)}_{\mu_2}\ldots\Gamma^{(+)}_{\mu_{2p}} & \\ 0 & \Gamma^{(-)}_{\mu_1}\Gamma^{(-)}_{\mu_2}\ldots\Gamma^{(-)}_{\mu_{2p}}\end{pmatrix} = (-1)^{n-p+1}\varepsilon_{\mu_1\mu_2\ldots\mu_{2p}\mu_{2p+1}\ldots\mu_{4n-1}}\begin{pmatrix}\Gamma^{(+)}_{\mu_{2p+1}}\Gamma^{(+)}_{\mu_{2p+2}}\ldots\Gamma^{(+)}_{\mu_{4n-1}} & 0\\ 0 & -\Gamma^{(-)}_{\mu_{2p+1}}\Gamma^{(-)}_{\mu_{2p+2}}\ldots\Gamma^{(-)}_{\mu_{4n-1}}\end{pmatrix}. \tag{2.126}$$

Right multiply $\Gamma^{(\pm)}_{4n} = 1$ on the r.h.s.:

$$\begin{pmatrix}\Gamma^{(+)}_{\mu_1}\Gamma^{(+)}_{\mu_2}\ldots\Gamma^{(+)}_{\mu_{2p}} & \\ 0 & \Gamma^{(-)}_{\mu_1}\Gamma^{(-)}_{\mu_2}\ldots\Gamma^{(-)}_{\mu_{2p}}\end{pmatrix} = (-1)^{n-p+1}\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}4n}\begin{pmatrix}\Gamma^{(+)}_{\mu_{2p+1}}\Gamma^{(+)}_{\mu_{2n+2}}\ldots\Gamma^{(+)}_{\mu_{4n-1}}\Gamma^{(+)}_{4n} & 0\\ 0 & -\Gamma^{(-)}_{\mu_{2p+1}}\Gamma^{(-)}_{\mu_{2n+2}}\ldots\Gamma^{(-)}_{\mu_{4n-1}}\Gamma^{(-)}_{4n}.\end{pmatrix} \tag{2.127}$$

Now we rewrite $\Gamma^{(\pm)}_{\mu}$ to the ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$ with using (2.109) and $\Gamma^{(+)}_i = -\Gamma^{(-)}_i$, and then we can include the index $4n$ to the anti-symmetric tensor ε with using $\Gamma^{(\pm)}_i\Gamma^{(\pm)}_{4n} = \Sigma^{(\mp)}_{i4n}$ and $\Gamma^{(\pm)}_{4n}\Gamma^{(\mp)}_i = -\Sigma^{(\pm)}_{4ni}$. Hence the index $4n$ in the r.h.s. case, we obtain the following relation for similar calculation of the basic order case:

$$\Sigma^{(\pm)}_{[\mu_1\mu_2}\ldots\Sigma^{(\pm)}_{\mu_{2p-1}\mu_{2p}]} = \pm(-1)^{n-p}2^{-2(n-p)}\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}\Sigma^{(\pm)}_{\mu_{2n+1}\mu_{2n+2}}\ldots\Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}}, \tag{2.128}$$

where μ_i is not summed. On the other hand, in the case of $4n$ in l.h.s., we right multiply $\Gamma_{\mu_{4n-1}}\ldots\Gamma_{\mu_{2n}}$ in the starting equation (2.124). Therefore (2.125) replace by

$$\Gamma_{\mu_1}\Gamma_{\mu_2}\ldots\Gamma_{\mu_{2n-1}} = (-1)^{n+1}\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\times(-1)^{-p}\Gamma_{\mu_{2n}}\Gamma_{\mu_{2n+2}}\ldots\Gamma_{\mu_{4n-1}}. \tag{2.129}$$

In this case, the factor of the l.h.s. becomes $(-1)^{2(2n-p)} = 1$ becase $2(2n-p)$ times product $\Gamma_i^2 = -1$ ($i = 2p,\ldots,4n-1$ and note summed). In the r.h.s., we take operate $\Gamma_{4n-1}\Gamma_{4n-2}\ldots\Gamma_{2p} \to (-1)^{2(2n-p)-1}\Gamma_{4n-2}\ldots\Gamma_{2p}\Gamma_{4n-1} \to (-1)^{2(2n-p)-1}(-1)^{2(2n-p)-2}\Gamma_{4n-2}\ldots\Gamma_{2p}\Gamma_{4n-2}\Gamma_{4n-1} \to \ldots$ thus the exponent of (-1) is $\sum_{i=1}^{2(2n-p)} i = 2(2n-p)^2 + (2n-p)$ times. Therefore the factor of the r.h.s. is $(-1)^{-p}$.

After that, we take same operate in the above, and then right multiply $\Gamma^{(\pm)}_{4n} = 1$ on the l.h.s. of corresponding equation for (2.126). Here we using $\varepsilon_{\mu_1\ldots\mu_{2p-1}\mu_{2p}\ldots\mu_{4n-1}4n} = \varepsilon_{\mu_1\ldots\mu_{2p-1}4n\mu_{2p}\ldots\mu_{4n-1}}$, we obtain (2.128) again. The equation (2.128) is not summed $\mu_{2n+1}\ldots\mu_{4n}$. Now, using the lemma 2.4.1, we can rewrite the equation to summed index notation:

$$\Sigma^{(\pm)}_{[\mu_1\mu_2}\ldots\Sigma^{(\pm)}_{\mu_{2p-1}\mu_{2p}]} = \pm\frac{(-1)^{n-p}}{2^{2(n-p)}\cdot(2(2n-p))!}\varepsilon_{\mu_1\mu_2\ldots\mu_{2p}\mu_{2p+1}\ldots\mu_{4n-1}\mu_{4n}}\Sigma^{(\pm)}_{\mu_{2p+1}\mu_{2p+2}}\ldots\Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} \tag{2.130}$$

$\mathcal{Q.E.D.}$

In the teorem 2.4.1, we take $p = n$ then obtain the following corollary which plays most importance of the $4n$-dimensional ADHM construction.

Corollary 2.4.1 (The (anti-)self-duality of the $4n$-dimensional ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$**).**
Here $\Sigma^{(\pm)}_{\mu\nu}$ is the ASD tensor which is defined by (2.8)*, then ASD tensor satisfies the* $4n$*-dimensional ASD relation:*

$$\Sigma^{(\pm)}_{[\mu_1\mu_2}\ldots\Sigma^{(\pm)}_{\mu_{2n-1}\mu_{2n}]} = \pm\frac{1}{(2n)!}\varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}\Sigma^{(\pm)}_{\mu_{2n+1}\mu_{2n+2}}\ldots\Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} \tag{2.131}$$

Remark 2.4.1. The orign of factor $2^{-2(n-p)}$ in (2.116) is the relation between the ASD tensor with the the (decomposed) Clifford algebra in (2.109). Thus we can show Hodge duality more clearly by changing the difinition of the ASD tensor (2.8). We define deformed ASD tensor $\tilde{\Sigma}^{(\pm)}$ by

$$\tilde{\Sigma}^{(+)}_{\mu\nu} := \frac{i}{2}\Sigma^{(+)}_{\mu\nu} = \frac{i}{2}\left(e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu\right),\quad \tilde{\Sigma}^{(-)}_{\mu\nu} := \frac{i}{2}\Sigma^{(-)}_{\mu\nu} = \frac{i}{2}\left(e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu\right), \tag{2.132}$$

where we multiply the imaginay to cancel numerator $(-1)^{n-p}$. Now the Hodge duality of ASD tensor (2.116) becomes

$$\tilde{\Sigma}^{(\pm)}_{[\mu_1\mu_2} \dots \tilde{\Sigma}^{(\pm)}_{\mu_{2p-1}\mu_{2p}]} = \pm \frac{1}{(2(2n-p))!} \varepsilon_{\mu_1\mu_2\dots\mu_{2p}\mu_{2p+1}\dots\mu_{4n-1}\mu_{4n}} \tilde{\Sigma}^{(\pm)}_{\mu_{2p+1}\mu_{2p+2}} \dots \tilde{\Sigma}^{(\pm)}_{\mu_{4n-1}\mu_{4n}}. \tag{2.133}$$

Although the equation becomes more simple when we use the deformed ASD tensor, other calculations become more complicated in generally. In this paper, we have been treat ASD instantons, namely the (anti-)self-duality case. In this case, the equation is same form for both ASD tensor difinition, hence we have been take the simple definition.

Finally, we will show the whole product of ASD tensor (2.13) which is used the charge density formula of ADHM data (2.50).

Lemma 2.4.2.

$$\Sigma^{(\pm)}_{12} \dots \Sigma^{(\pm)}_{(4n-1)4n} = \pm(-1)^n 2^{2n} \mathbf{1}, \tag{2.134}$$

where $\mathbf{1}$ *is identity element of the decomposed Clifford algebra* $C\ell^{(\pm)}_{4n-1}(K)$*.*

Proof. This proof start from the following relation again.

$$\Gamma_1\Gamma_2 \dots \Gamma_{4n-1} = (-1)^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.118}$$

For $\Gamma_i = \Gamma^{(+)}_i \oplus \Gamma^{(-)}_i$,

$$\begin{pmatrix} \Gamma^{(+)}_1\Gamma^{(+)}_2 \dots \Gamma^{(+)}_{4n-1} & 0 \\ 0 & \Gamma^{(-)}_1\Gamma^{(-)}_2 \dots \Gamma^{(-)}_{4n-1} \end{pmatrix} = (-1)^{n-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.135}$$

We right multiply the $\Gamma^{(\pm)}_{4n} = 1$ on the l.h.s., and then rewrite $\Gamma^{(\pm)}_\mu$ to the ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$ with using (2.109) and $\Gamma^{(+)}_i = -\Gamma^{(-)}_i$:

$$\begin{aligned} \begin{pmatrix} \Gamma^{(+)}_1\Gamma^{(+)}_2 \dots \Gamma^{(+)}_{4n-1}\Gamma^{(+)}_{4n} & 0 \\ 0 & \Gamma^{(-)}_1\Gamma^{(-)}_2 \dots \Gamma^{(-)}_{4n-1}\Gamma^{(-)}_{4n} \end{pmatrix} &= \begin{pmatrix} (-1)^{2n-1}\Gamma^{(+)}_1\Gamma^{(-)}_2 \dots \Gamma^{(+)}_{4n-1}\Gamma^{(-)}_{4n} & 0 \\ 0 & (-1)^{2n-1}\Gamma^{(-)}_1\Gamma^{(+)}_2 \dots \Gamma^{(-)}_{4n-1}\Gamma^{(+)}_{4n} \end{pmatrix} \\ &= \begin{pmatrix} (-1)\left(\frac{1}{2}\right)^{2n} \Sigma^{(+)}_{12} \dots \Sigma^{(+)}_{(4n-1)(4n)} & 0 \\ 0 & (-1)\left(\frac{1}{2}\right)^{2n} \Sigma^{(-)}_{12} \dots \Sigma^{(-)}_{(4n-1)(4n)} \end{pmatrix}, \end{aligned} \tag{2.136}$$

thus we obtain

$$(-1)\Sigma^{(\pm)}_{12} \dots \Sigma^{(\pm)}_{(4n-1)4n} = \pm(-1)^{n-1} 2^{2n} \mathbf{1}. \tag{2.137}$$

Q.E.D.

Using this lemma, we obtain the whole product relation (2.13).

Theorem 2.4.2 (The whole product of the ASD tensor)**.**

$$\Sigma^{(\pm)}_{\mu_1\mu_2} \dots \Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} = \pm(-1)^n \varepsilon_{\mu_1\mu_2\dots\mu_{4n-1}\mu_{4n}} 2^{2n} \mathbf{1}, \tag{2.138}$$

where $\mu_i \neq \mu_j$*.*

Proof. First we recall the following relation:

$$\Gamma_{i_1}\Gamma_{i_2} \dots \Gamma_{i_{4n-1}} = \varepsilon_{i_1 i_2 \dots i_{4n-1}} \Gamma_1\Gamma_2 \dots \Gamma_{4n-1}, \tag{2.139}$$

where $i_p \in \{1, \dots, 4n-1\}$ and $i_p \neq i_q$. We right mutiply $\Gamma_\# = 1(\ \# = 4n)$ on the both sides, and then we obtain the folloing with using same calculation of the proof for above lemma 2.4.2:

$$\Sigma^{(\pm)}_{i_1 i_2} \dots \Sigma^{(\pm)}_{i_{4n-1}\#} = \pm(-1)^n \varepsilon_{i_1 i_2 \dots i_{4n-1}} \mathbf{1}, \tag{2.140}$$

Next we conside the case that the position of the index : $\# = 4n$ is arbitrary. Now we recall the following relations:

$$\Sigma^{(\pm)}_{i\#} = -\Sigma^{(\pm)}_{\# i}, \tag{2.141a}$$

$$\Sigma^{(\pm)}_{ij}\Sigma^{(\pm)}_{\# k} = 2\Gamma^{(\pm)}_i\Gamma^{(\mp)}_j \cdot 2\mathbf{1}\Gamma^{(\mp)}_k = -2\Gamma^{(\pm)}_i\mathbf{1} \cdot 2\Gamma^{(\pm)}_j\Gamma^{(\mp)}_k = -\Sigma^{(\pm)}_{i\#}\Sigma^{(\pm)}_{jk}. \quad \because). \ \Gamma^{(\pm)}_i = -\Gamma^{(\mp)}_i. \tag{2.141b}$$

From these relations, the factor (-1) is multiplied whenever to shift the index #. This means just that $\Sigma^{(\pm)}_{\mu_1\mu_2} \ldots \Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}}$ is completely antisymmetric tensor. Therefore we can include the index # in the completely antisymmetric tensor ε, we rewrite (2.140) to

$$\Sigma^{(\pm)}_{\mu_1\mu_2} \ldots \Sigma^{(\pm)}_{\mu_{4n-1}\mu_{4n}} = \pm(-1)^n 2^{2n} \varepsilon_{\mu_1\mu_2\ldots\mu_{4n-1}\mu_{4n}}\mathbf{1} \tag{2.142}$$

$\mathcal{Q.E.D.}$

2.4.3 Proof that the existence of the inverse $E^{(a)}_k$ $(a = 1, 2)$

We show again the ADHM constraints for convenience.

$$\Delta^\dagger\Delta = \mathbf{1}_{2^{2n-1}} \otimes E^{(1)}_k, \tag{2.143}$$

$$C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C = \mathbf{1}_{2^{2n-1}} \otimes E^{(2)}_k. \tag{2.144}$$

Lets the Weyl operator satisfy the non-degeneracy condition:

$$\text{rank}\ \Delta = 2^{2n-1}k. \tag{2.145}$$

First, we show the existence of the inverse $E^{(1)}_k$. Because of the property of the rank: rank Δ = rank $\Delta^\dagger\Delta = 2^{2n-1}k$. We recall that Δ is the $(N+2^{2n-1}k)\times 2^{2n-1}k$ matrix thus dim $\Delta^\dagger\Delta = 2^{2n-1}k$, and we use the rank-nullity theorem: dim $\Delta^\dagger\Delta$ = rank $\Delta^\dagger\Delta$ + Ker $\Delta^\dagger\Delta$, then we obtain

$$\text{Ker}\ \Delta^\dagger\Delta = 0, \tag{2.146}$$

where Ker A means the dimension of the kernel of A. If the kernel dimension of the matrix A is zero then the inverse matrix of A is existence, hence there is $(\Delta^\dagger\Delta)^{-1}$ for (2.146). This is just the assurance of the existence of the inverse $E^{(1)}_k$.

Next, we will prove the existence of the inverse $E^{(2)}_k$. The Weyl operator Δ contains the coordinate parameter x, therefore the non-degeneracy condition (2.145) holds for all x [4]. Because of this fact and $\Delta = C(x \otimes \mathbf{1}_k) + D$, we obtain

$$\text{rank}\ \Delta(\infty) = \text{rank}\ C(x \otimes \mathbf{1}_k) = 2^{2n-1}k \tag{2.147}$$

Now we recall that $x \otimes \mathbf{1}_k = x^\mu e_\mu \otimes \mathbf{1}_k$ is the $2^{2n-1}k \times 2^{2n-1}k$ invertible matrix, since we can give the inverse of x as $x^{-1} = \frac{x^\mu}{||x||^2}e^\dagger_\mu$ explicitly. Hence,

$$\text{rank}\ C = \text{rank}\ C(x \otimes \mathbf{1}_k) = 2^{2n-1}k. \tag{2.148}$$

We can also obtain the rank of D: rank D = rank $\Delta(0) = 2^{2n-1}k$. These facts give that rank $\Delta = 2^{2n-1}k \Rightarrow$ rank C = rank $D = 2^{2n-1}k$, and the inverse fact: rank C = rank $D = 2^{2n-1}k \Rightarrow$ rank $\Delta = 2^{2n-1}k$ is trivial. Therefore we obtain rank $\Delta = 2^{2n-1}k \iff$ rank C = rank $D = 2^{2n-1}k$. We take the $2^{2n-1}k \times 2^{2n-1}k$ matrix $C^\dagger\Delta$ to the same situations as (2.147) and (2.148), and we use the expansion $C^\dagger\Delta = C^\dagger C(x \otimes \mathbf{1}_k) + C^\dagger D$ then

$$\text{rank}\ C^\dagger\Delta(x) = \text{rank}\ C^\dagger\Delta(\infty) = \text{rank}\ C^\dagger C(x \otimes \mathbf{1}_k) = \text{rank}\ C^\dagger C = \text{rank}\ C = 2^{2n-1}k. \tag{2.149}$$

For this reasion and the rank-nullity theorem, we obtain Ker $C^\dagger\Delta$ = Ker $\Delta^\dagger C = 0$. This means that the maps $C^\dagger\Delta : \mathbb{C}^{2^{2n-1}k} \to \mathbb{C}^{2^{2n-1}k}$ and $\Delta^\dagger C : \mathbb{C}^{2^{2n-1}k} \to \mathbb{C}^{2^{2n-1}k}$ are bijective, and the map $(\Delta^\dagger\Delta)^{-1} : \mathbb{C}^{2^{2n-1}k} \to \mathbb{C}^{2^{2n-1}k}$ is also bijective from the above proof. Therefore, using the bijective map composition, we find that the map $C^\dagger\Delta \circ (\Delta^\dagger\Delta)^{-1} \circ \Delta^\dagger C : \mathbb{C}^{2^{2n-1}k} \to \mathbb{C}^{2^{2n-1}k}$ becomes bijective. If a map is bijective then the existence of the inverse map, thus the matrix $C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C$ is invertible. Therefore we have shown the existence of the inverse $E^{(2)}_k$.

[4]Technically $\Delta(x)$ does not have to satisfy the non-degeneracy condition at the instantons positions.

2.4.4 The approximate charge density of the 't Hooft $k = 2, 3$ instantons

Here we lead to the approximate charge density of the 't Hooft $k = 2, 3$ instantons (2.77).

For latter convenience, we show the following relations with using $e^\dagger_\mu e_\nu = \Sigma^{(+)}_{\mu\nu}/2 + \delta_{\mu\nu}$.

$$2\left(\tilde{x}^\mu_a e^\dagger_\nu - \tilde{x}^\nu_a e^\dagger_\mu\right)\tilde{x}_b = \tilde{x}^\mu_a \tilde{x}^\rho_b \Sigma^{(+)}_{\nu\rho} - \tilde{x}^\nu_a \tilde{x}^\rho_a \Sigma^{(+)}_{\mu\rho}, \tag{2.150a}$$

$$\tilde{x}^\dagger_a \tilde{x}_b + \tilde{x}^\dagger_b \tilde{x}_a = \tilde{x}^\mu_a \tilde{x}^\nu_b \left(e^\dagger_\mu e_\nu + e^\dagger_\nu e_\mu\right) = \tilde{x}^\mu_a \tilde{x}^\nu_b \left(\Sigma^{(+)}_{\mu\nu}/2 + \Sigma^{(+)}_{\nu\mu}/2 + 2\delta_{\mu\nu}\right) = 2\tilde{x}^\mu_a \tilde{x}^\mu_b. \tag{2.150b}$$

$k = 2$ approximate chage density

We first calculate $1/\phi$:

$$\frac{1}{\phi} = \frac{1}{1 + \frac{\lambda_1^2}{||\tilde{x}_1||^2} + \frac{\lambda_2^2}{||\tilde{x}_2||^2}} = \frac{||\tilde{x}_1||^2 ||\tilde{x}_2||^2}{||\tilde{x}_1||^2||\tilde{x}_2||^2 + \lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2}. \tag{2.151}$$

$$\begin{aligned}
f^{\mu\nu} e^\dagger_\mu e_\nu &= \begin{pmatrix} \frac{\lambda_1 \tilde{x}^\mu_1}{||\tilde{x}_1||^2} & \frac{\lambda_2 \tilde{x}^\mu_2}{||\tilde{x}_2||^2} \end{pmatrix} \frac{1}{\left(\lambda_1^2 + ||\tilde{x}_1||^2\right)\left(\lambda_2^2 + ||\tilde{x}_2||^2\right) - \lambda_1^2\lambda_2^2} \begin{pmatrix} \lambda_2^2 + ||\tilde{x}_2||^2 & -\lambda_1\lambda_2 \\ -\lambda_1\lambda_2 & \lambda_1^2 + ||\tilde{x}_1||^2 \end{pmatrix} \begin{pmatrix} \frac{\lambda_1 \tilde{x}^\nu_1}{||\tilde{x}_1||^2} \\ \frac{\lambda_2 \tilde{x}^\nu_2}{||\tilde{x}_2||^2} \end{pmatrix} e^\dagger_\mu e_\nu \\
&= \frac{1}{\lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2} \left(\frac{\lambda_1^2\left(\lambda_2^2 + ||\tilde{x}_2||^2\right)\tilde{x}^\mu_1 \tilde{x}^\nu_1}{||\tilde{x}_1||^4} + \frac{-\lambda_1^2\lambda_2^2 \tilde{x}^\mu_2 \tilde{x}^\nu_1}{||\tilde{x}_1||^2||\tilde{x}_2||^2} + \frac{-\lambda_1^2\lambda_2^2 \tilde{x}^\mu_1 \tilde{x}^\nu_2}{||\tilde{x}_1||^2||\tilde{x}_2||^2} + \frac{\lambda_2^2\left(\lambda_1^2 + ||\tilde{x}_1||^2\right)\tilde{x}^\mu_2 \tilde{x}^\nu_2}{||\tilde{x}_2||^4} \right) e^\dagger_\mu e_\nu \\
&= \frac{1}{\lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2} \left(\frac{\lambda_1^2\left(\lambda_2^2 + ||\tilde{x}_2||^2\right)}{||\tilde{x}_1||^2} \mathbf{1}_{2^{2n-1}} - \frac{\lambda_1^2\lambda_2^2}{||\tilde{x}_1||^2||\tilde{x}_2||^2} \underbrace{\left(\tilde{x}^\dagger_1 \tilde{x}_2 + \tilde{x}^\dagger_2 \tilde{x}_1\right)}_{=2\tilde{x}^\mu_1 \tilde{x}^\mu_2 \mathbf{1}_2^{2n-1}} + \frac{\lambda_2^2\left(\lambda_1^2 + ||\tilde{x}_1||^2\right)}{||\tilde{x}_2||^2} \mathbf{1}_{2^{2n-1}} \right) \\
&= \frac{\lambda_1^2 ||\tilde{x}_2||^4 + \lambda_2^2 ||\tilde{x}_1||^4 + \lambda_1^2\lambda_2^2\left(||\tilde{x}_1||^2 + ||\tilde{x}_2||^2 - 2\tilde{x}^\mu_1 \tilde{x}^\mu_2\right)}{\left(\lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2\right) ||\tilde{x}_1||^2||\tilde{x}_2||^2} \mathbf{1}_{2^{2n-1}}
\end{aligned} \tag{2.152}$$

Therefore the $k = 2$ approximate charge density becomes

$$\begin{aligned}
Q^{(k=2)} &= -\frac{(4n)!}{\phi^{2n}} \mathrm{Tr}\left(\frac{\lambda_1^2 ||\tilde{x}_2||^4 + \lambda_2^2 ||\tilde{x}_1||^4 + \lambda_1^2\lambda_2^2\left(||\tilde{x}_1||^2 + ||\tilde{x}_2||^2 - 2\tilde{x}^\mu_1 \tilde{x}^\mu_2\right)}{\left(\lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2\right) ||\tilde{x}_1||^2||\tilde{x}_2||^2} \mathbf{1}_{2^{2n-1}} \right)^{2n} \\
&= -(4n)! \cdot 2^{2n-1} \left(\frac{\lambda_1^2 ||\tilde{x}_2||^4 + \lambda_2^2 ||\tilde{x}_1||^4 + \lambda_1^2\lambda_2^2\left(||\tilde{x}_1||^2 + ||\tilde{x}_2||^2 - 2\tilde{x}^\mu_1 \tilde{x}^\mu_2\right)}{\left(\lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2\right)^2} \right)^{2n}.
\end{aligned} \tag{2.153}$$

$k = 3$ approximate charge density

We first calculate the inverce matrix of $E^{(1)}_k$:

$$\begin{aligned}
(E^{(1)}_k)^{-1} &= \begin{pmatrix} \lambda_1^2 + ||\tilde{x}_1||^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 + ||\tilde{x}_2||^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 + ||\tilde{x}_3||^2 \end{pmatrix}^{-1} \\
&= \frac{1}{\left(\lambda_1^2 + ||\tilde{x}_1||^2\right)\left(\lambda_2^2 + ||\tilde{x}_2||^2\right)\left(\lambda_3^2 + ||\tilde{x}_3||^2\right) + 2\lambda_1^2\lambda_2^2\lambda_3^2 - \lambda_1^2\lambda_2^2\left(\lambda_3^2 + ||\tilde{x}_3||^2\right) - \lambda_2^2\lambda_3^2\left(\lambda_1^2 + ||\tilde{x}_1||^2\right) - \lambda_1^2\lambda_3^2\left(\lambda_2^2 + ||\tilde{x}_2||^2\right)} \\
&\quad \times \begin{pmatrix} \left(\lambda_2^2 + ||\tilde{x}_2||^2\right)\left(\lambda_3^2 + ||\tilde{x}_3||^2\right) - \lambda_2^2\lambda_3^2 & \lambda_1\lambda_2\lambda_3^2 - \lambda_1\lambda_2\left(\lambda_3^2 + ||\tilde{x}_3||^2\right) & \lambda_1\lambda_2^2\lambda_3 - \lambda_1\lambda_3\left(\lambda_2^2 + ||\tilde{x}_2||^2\right) \\ \lambda_1\lambda_2^2\lambda_3 - \lambda_1\lambda_2\left(\lambda_3^2 + ||\tilde{x}_3||^2\right) & \left(\lambda_1^2 + ||\tilde{x}_1||^2\right)\left(\lambda_3^2 + ||\tilde{x}_3||^2\right) - \lambda_1^2\lambda_3^2 & \lambda_1^2\lambda_2\lambda_3 - \lambda_2\lambda_3\left(\lambda_1^2 + ||\tilde{x}_1||^2\right) \\ \lambda_1\lambda_2^2\lambda_3 - \lambda_1\lambda_3\left(\lambda_2^2 + ||\tilde{x}_2||^2\right) & \lambda_1^2\lambda_2\lambda_3 - \lambda_2\lambda_3\left(\lambda_1^2 + ||\tilde{x}_1||^2\right) & \left(\lambda_1^2 + ||\tilde{x}_1||^2\right)\left(\lambda_2^2 + ||\tilde{x}_2||^2\right) - \lambda_1^2\lambda_2^2 \end{pmatrix} \\
&= \overline{\gamma} \begin{pmatrix} \lambda_2^2 ||\tilde{x}_3||^2 + \lambda_3^2 ||\tilde{x}_2||^2 + ||\tilde{x}_2||^2||\tilde{x}_3||^2 & -\lambda_1\lambda_2 ||\tilde{x}_3||^2 & -\lambda_1\lambda_3 ||\tilde{x}_2||^2 \\ -\lambda_1\lambda_2 ||\tilde{x}_3||^2 & \lambda_1^2 ||\tilde{x}_3||^2 + \lambda_3^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_3||^2 & -\lambda_2\lambda_3 ||\tilde{x}_1||^2 \\ -\lambda_1\lambda_3 ||\tilde{x}_2||^2 & -\lambda_2\lambda_3 ||\tilde{x}_1||^2 & \lambda_1^2 ||\tilde{x}_2||^2 + \lambda_2^2 ||\tilde{x}_1||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2 \end{pmatrix},
\end{aligned} \tag{2.154}$$

$$\text{where } \overline{\gamma} := \frac{1}{\lambda_1^2 ||\tilde{x}_2||^2||\tilde{x}_3||^2 + \lambda_2^2 ||\tilde{x}_1||^2||\tilde{x}_3||^2 + \lambda_3^2 ||\tilde{x}_1||^2||\tilde{x}_2||^2 + ||\tilde{x}_1||^2||\tilde{x}_2||^2||\tilde{x}_3||^2}.$$

ϕ becomes

$$\phi = 1 + \frac{\lambda_1^2}{\|\tilde{x}_1\|^2} + \frac{\lambda_2^2}{\|\tilde{x}_2\|^2} + \frac{\lambda_3^2}{\|\tilde{x}_3\|^2} = \frac{\lambda_1^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2 + \lambda_2^2\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_1\|^2\tilde{x}_2\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}. \tag{2.155}$$

Using above results,

$$\begin{aligned}
f^{\mu\nu}e_\mu^\dagger e_\nu &= \begin{pmatrix} \frac{\lambda_1\tilde{x}_1^\mu}{\|\tilde{x}_1\|^2} & \frac{\lambda_2\tilde{x}_2^\mu}{\|\tilde{x}_2\|^2} & \frac{\lambda_3\tilde{x}_3^\mu}{\|\tilde{x}_3\|^2}\end{pmatrix} \overline{\gamma} \begin{pmatrix} \lambda_2^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_2\|^2 + \|\tilde{x}_2\|^2\|\tilde{x}_3\|^2 & -\lambda_1\lambda_2\|\tilde{x}_3\|^2 & -\lambda_1\lambda_3\|\tilde{x}_2\|^2 \\ -\lambda_1\lambda_2\|\tilde{x}_3\|^2 & \lambda_1^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_1\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_3\|^2 & -\lambda_2\lambda_3\|\tilde{x}_1\|^2 \\ -\lambda_1\lambda_3\|\tilde{x}_2\|^2 & -\lambda_2\lambda_3\|\tilde{x}_1\|^2 & \lambda_1^2\|\tilde{x}_2\|^2 + \lambda_2^2\|\tilde{x}_1\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2 \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \frac{\lambda_1\tilde{x}_1^\nu}{\|\tilde{x}_1\|^2} \\ \frac{\lambda_2\tilde{x}_2^\nu}{\|\tilde{x}_2\|^2} \\ \frac{\lambda_3\tilde{x}_3^\nu}{\|\tilde{x}_3\|^2} \end{pmatrix} e_\mu^\dagger e_\nu \\
&= \overline{\gamma}\Bigg(\frac{\lambda_1^2\left(\lambda_2^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_2\|^2 + \|\tilde{x}_2\|^2\|\tilde{x}_3\|^2\right)\tilde{x}_1^\mu\tilde{x}_1^\nu}{\|\tilde{x}_1\|^4} + \frac{-\lambda_1^2\lambda_2^2\|\tilde{x}_3\|^2\tilde{x}_2^\mu\tilde{x}_1^\nu}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2} + \frac{-\lambda_1^2\lambda_3^2\|\tilde{x}_2\|^2\tilde{x}_3^\mu\tilde{x}_1^\nu}{\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2} \\
&\quad + \frac{-\lambda_1^2\lambda_2^2\|\tilde{x}_3\|^2\tilde{x}_1^\mu\tilde{x}_2^\nu}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2} + \frac{\lambda_2^2\left(\lambda_1^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_1\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_3\|^2\right)\tilde{x}_2^\mu\tilde{x}_2^\nu}{\|\tilde{x}_2\|^4} + \frac{-\lambda_2^2\lambda_3^2\|\tilde{x}_1\|^2\tilde{x}_3^\mu\tilde{x}_2^\nu}{\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2} \\
&\quad + \frac{-\lambda_1^2\lambda_3^2\|\tilde{x}_2\|^2\tilde{x}_1^\mu\tilde{x}_3^\nu}{\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2} + \frac{-\lambda_2^2\lambda_3^2\|\tilde{x}_1\|^2\tilde{x}_2^\mu\tilde{x}_3^\nu}{\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2} + \frac{\lambda_3^2\left(\lambda_1^2\|\tilde{x}_2\|^2 + \lambda_2^2\|\tilde{x}_1\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\right)\tilde{x}_3^\mu\tilde{x}_3^\nu}{\|\tilde{x}_3\|^4}\Bigg)e_\mu^\dagger e_\nu \\
&= \overline{\gamma}\Bigg(\frac{\lambda_1^2\|\tilde{x}_2\|^4\|\tilde{x}_3\|^4 + \lambda_2^2\|\tilde{x}_1\|^4\|\tilde{x}_3\|^4 + \lambda_3^2\|\tilde{x}_1\|^4\|\tilde{x}_2\|^4}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}\mathbf{1}_{2^{2n-1}} + \frac{\lambda_1^2\lambda_2^2\|\tilde{x}_3\|^4}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}\left((\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2)\mathbf{1}_{2^{2n-1}} - \tilde{x}_2^\dagger\tilde{x}_1 - \tilde{x}_1^\dagger\tilde{x}_2\right) \\
&\quad + \frac{\lambda_2^2\lambda_3^2\|\tilde{x}_1\|^4}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}\left((\|\tilde{x}_2\|^2 + \|\tilde{x}_3\|^2)\mathbf{1}_{2^{2n-1}} - \tilde{x}_3^\dagger\tilde{x}_2 - \tilde{x}_2^\dagger\tilde{x}_3\right) + \frac{\lambda_1^2\lambda_3^2\|\tilde{x}_2\|^4}{\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}\left((\|\tilde{x}_1\|^2 + \|\tilde{x}_3\|^2)\mathbf{1}_{2^{2n-1}} - \tilde{x}_3^\dagger\tilde{x}_1 - \tilde{x}_1^\dagger\tilde{x}_3\right)\Bigg) \\
&= \frac{1}{\left(\lambda_1^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2 + \lambda_2^2\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2\right)\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2}\times \\
&\quad \Bigg(\lambda_1^2\|\tilde{x}_2\|^4\|\tilde{x}_3\|^4 + \lambda_2^2\|\tilde{x}_1\|^4\|\tilde{x}_3\|^4 + \lambda_3^2\|\tilde{x}_1\|^4\|\tilde{x}_2\|^4 + \lambda_1^2\lambda_2^2\|\tilde{x}_3\|^4\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 - 2\tilde{x}_1^\mu\tilde{x}_2^\mu\right) \\
&\qquad + \lambda_2^2\lambda_3^2\|\tilde{x}_1\|^4\left(\|\tilde{x}_2\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}_2^\mu\tilde{x}_3^\mu\right) + \lambda_1^2\lambda_3^2\|\tilde{x}_2\|^4\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}_1^\mu\tilde{x}_3^\mu\right)\Bigg)\mathbf{1}_{2^{2n-1}}.
\end{aligned} \tag{2.156}$$

Therefore we obtain the $k = 3$ approximate charge density as

$$\begin{aligned}
Q^{(k=3)} &= \frac{-(4n)!}{\phi^{2n}}\mathrm{Tr}\left[f^{\mu\nu}e_\mu^\dagger e_\nu\right]^{2n} \\
&= -(4n)! \cdot 2^{2n-1}\Bigg[\gamma\Big(\lambda_1^2\|\tilde{x}_2\|^4\|\tilde{x}_3\|^4 + \lambda_2^2\lambda_3^2\|\tilde{x}_1\|^4\left(\|\tilde{x}_2\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}_2^\mu\tilde{x}_3^\mu\right) \\
&\qquad + \lambda_2^2\|\tilde{x}_1\|^4\|\tilde{x}_3\|^4 + \lambda_1^2\lambda_3^2\|\tilde{x}_2\|^4\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}_1^\mu\tilde{x}_3^\mu\right) \\
&\qquad + \lambda_3^2\|\tilde{x}_1\|^4\|\tilde{x}_2\|^4 + \lambda_1^2\lambda_2^2\|\tilde{x}_3\|^4\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 - 2\tilde{x}_1^\mu\tilde{x}_2^\mu\right)\Big)\Bigg]^{2n},
\end{aligned} \tag{2.157}$$

where

$$\gamma = \frac{1}{\left(\lambda_1^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2 + \lambda_2^2\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2\right)^2}. \tag{2.158}$$

2.5 Clifford algebra and $4n$-dimensional ASD tensor

We showed that the $4n$-dimensional ASD basis (tensor) which played the central roles of the ADHM construcion was constructed from the $(4n-1)$-dimensional Clifford algebra. In this section, we consider the Clifford algebra and the $4n$-dimensional ASD

$m_+ - m_-$ mod 8	$\tilde{\omega}^2$	$C\ell_m(\mathbb{R})$	$m_+ - m_-$ mod 8	$\tilde{\omega}^2$	$C\ell_m(\mathbb{R})$
0	+	$\mathrm{GL}(2^{m/2}, \mathbb{R})$	1	+	$\mathrm{GL}(2^{(m-1)/2}, \mathbb{R}) \oplus \mathrm{GL}(2^{(m-1)/2}, \mathbb{R})$
2	−	$\mathrm{GL}(2^{m/2}, \mathbb{R})$	3	−	$\mathrm{GL}(2^{(m-1)/2}, \mathbb{C})$
4	+	$\mathrm{GL}(2^{(m-2)/2}, \mathbb{H})$	5	+	$\mathrm{GL}(2^{(m-3)/2}, \mathbb{H}) \oplus \mathrm{GL}(2^{(m-3)/2}, \mathbb{H})$
6	−	$\mathrm{GL}(2^{(m-2)/2}, \mathbb{H})$	7	−	$\mathrm{GL}(2^{(m-1)/2}, \mathbb{C})$

Table 2.1: The matrix rings $\mathrm{GL}(N; K)$ which are isomorphic to the real Clifford algebra $C\ell_{m_+,m_-}(\mathbb{R})$. Here $m = m_+ + m_-$, N is the matrix size and the symbol $\mathbb{H}$ means the quaternion.

m mod 2	$C\ell_m(\mathbb{C})$
0	$\mathrm{GL}(2^{2m}, \mathbb{C})$
1	$\mathrm{GL}(2^{2m}, \mathbb{C}) \oplus \mathrm{GL}(2^{2m}, \mathbb{C})$

Table 2.2: The matrix rings $\mathrm{GL}(N; K)$ which are isomorphic to the complex Clifford algebra $C\ell_m(\mathbb{C})$. Note that $\tilde{\omega}$ is same situation as the real one.

basis in more detail. In particular, we will give the explicit representations of the ASD basis by using the matrix representation of the Clifford algebra.

Mathematically, a Clifford algebra is defined as an unital associative algebra that contains and is generated by a vector space V over a field K, where V is equipped with a quadratic form $Q : V \to K$. The Clifford algebra $C\ell(V, Q)$ is the most general algebra generated by V subject to the condition

$$^{\forall}v \in V, \;\; v^2 = Q(v)1, \tag{2.159}$$

where the product on the left is that of the algebra, and the 1 denotes its multiplicative identity. The most important Clifford algebras are those over real and complex vector spaces with nondegenerate quadratic forms.

Every nondegenerate quadratic form on a finite-dimensional space is equivalent ot the standard diagonal form:

$$Q(v) = v_1^2 + \cdots + v_{m_+}^2 - v_{m_++1}^2 - \cdots - v_{m_++m_-}^2, \tag{2.160}$$

where $m = m_+ + m_-$ is the dimension of the vector space. The pair of integers (m_+, m_-) is called the signature of the quadratic form. The real vector space with this quadratic form is often denoted $\mathbb{R}^{m_+,m_-}$, and then the Clifford algebra on $\mathbb{R}^{m_+,m_-}$ is denoted $C\ell_{m_+,m_-}(\mathbb{R})$ which is called as "the real Clifford algebra". For simplify, we take a standard basis $\{\Gamma_i\}$ for $\mathbb{R}^{m_+,m_-}$ which is consists of $m = m_+ + m_-$ mutually orthogonal vector, m_+ of which square to $+1$ and m_- of which square to -1. In this basis, the algebra $C\ell_{m_+,m_-}(\mathbb{R})$ have m_+ vectors that square to $+1$ and m_- vectors that square to -1. The symbol $C\ell_m(\mathbb{R})$ means either $C\ell_{m,0}(\mathbb{R})$ or $C\ell_{0,m}(\mathbb{R})$, we take the case that all signatures are negatives: $C\ell_m(\mathbb{R}) \equiv C\ell_{0,m}(\mathbb{R})$ in this paper.

On the other hand, we can consider a complex vector space instead of the real vector space. In this case, the standard diagonal form on the complex vector space is given by

$$Q(v) = v_1^2 + v_2^2 + \cdots + v_m^2, \tag{2.161}$$

where m is the dimension of the vector space. In the complex space, the signature of the quadratic form is not difference because we can take the imaginary product of elements $v_i \mapsto iv_i$ anytime. Therefore, when we consider up to isomorphism, there is only one nondegenerate Clifford algebra for each dimension m. We denote the Clifford algebra on $\mathbb{C}^m$ with the standard quadratic form by $C\ell_m(\mathbb{C})$, call "the complex Clifford algebra".

A particular importance of the real and the complex Clifford algebra is that each of the algebras is isomorphic to a full matrix ring with entries from $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Furthermore this isomorphism has the periodicity to m, known as "Bott periodicity", and we can completely classify the real and complex Clifford algebras with using this periodicity. Here we define $\tilde{\omega}$ as

$$\tilde{\omega} \coloneqq \Gamma_1 \ldots \Gamma_m \Longrightarrow \tilde{\omega}^2 = (\Gamma_1 \ldots \Gamma_m)(\Gamma_1 \ldots \Gamma_m), \tag{2.162}$$

where $\Gamma_i \in C\ell_m(K)$. For the Bott periodicity theorem [38], the real/complex Clifford algebra exhibit an eightfold/twofold periodicity (Table 2.1/2.2).

For convenience in this paper, we will organize the $(4n-1)$-dimensional Clifford algebras again as following. Elements of the real/complex Clifford algebra $\Gamma_i \in C\ell_m(K)$ ($K = \mathbb{R}/\mathbb{C}$) satisfying the relation:

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = -2\delta_{ij}, \tag{2.163}$$

4n-dim. mod 8	$C\ell_{4n-1}(\mathbb{C})$	$C\ell_{4n-1}(\mathbb{R})$
4	$\mathrm{GL}(2^{2n-1};\mathbb{C})\oplus\mathrm{GL}(2^{2n-1};\mathbb{C})$	$\mathrm{GL}(2^{2n-2};\mathbb{H})\oplus\mathrm{GL}(2^{2n-2};\mathbb{H})$
8	$\mathrm{GL}(2^{2n-1};\mathbb{C})\oplus\mathrm{GL}(2^{2n-1};\mathbb{C})$	$\mathrm{GL}(2^{2n-1};\mathbb{R})\oplus\mathrm{GL}(2^{2n-1};\mathbb{R})$

Table 2.3: The matrix rings $\mathrm{GL}(N;K)$ which are isomorphic to the $(4n-1)$-dimensional complex (real) Clifford algebra $C\ell_{4n-1}(\mathbb{C}(\mathbb{R}))$. Here N is the matrix size and the symbol $\mathbb{H}$ means the quaternion.

where the indices i, j run from 1 to m. In $m = 4n-1$ dimensions [5], we can introduce the chirality element ω as

$$\omega = (-1)^{\lfloor (m+5)/4\rfloor}\Gamma_1\Gamma_2\ldots\Gamma_m, \quad \Gamma_i \in C\ell_m(\mathbb{R}), \tag{2.164a}$$

$$\omega = i^{\lfloor (m+5)/2\rfloor}\Gamma_1\Gamma_2\ldots\Gamma_m, \quad \Gamma_i \in C\ell_m(\mathbb{C}), \tag{2.164b}$$

where the symbol $\lfloor x\rfloor$ is the floor function (for example: $\lfloor 2.8\rfloor = 2, \lfloor 3\rfloor = 3$). Here we define the overall factor of the chirality element ω for later convenience. For the Bott periodicity theorem (see Tabel 2.1 and Table 2.2), we can decompose the $(4n-1)$-dimensional Clifford algebra by using the chirality element. The projection operator is defined by

$$P_\pm = \frac{1}{2}(1 \pm \omega). \tag{2.165}$$

Using $P_\pm$, we can decompose the Clifford algebra as

$$C\ell_m(K) = C\ell_m^{(+)}(K) \oplus C\ell_m^{(-)}(K), \tag{2.166}$$

where $C\ell_m^{(\pm)}(K)$ are defined by elements in $C\ell_m(K)$ projected by $P_\pm$. We call $C\ell_m^{(\pm)}(K)$ "the decomposed Clifford algebra". Now we choose the elements of the decomposed Clifford algebra $\Gamma_i^{(\pm)} \in C\ell_m^{(\pm)}(K)$ that satisfy the relation $\Gamma_i^{(+)} = -\Gamma_i^{(-)}$.

Note that the elements of the decomposed Clifford algebra $\Gamma_i^{(\pm)} \in C\ell_m^{(\pm)}(K)$ satisfy the relation $\{\Gamma_i^{(\pm)}, \Gamma_j^{(\pm)}\} = -2\delta_{ij}$, but $\Gamma_i^{(\pm)}$ are not elements of the Clifford algebra. Because the elements of the decomposed Clifford algebra are not the algebraic generators. The algebraic generators have the property that each element of the algebra is not produced by a product of other elements, that is $e_ie_j\cdots \neq e_t$ where $e_i, e_j, \ldots, e_t \in Q(K)$ and $Q(K)$ is an algebra on the field K. The elements of the Clifford algebra Γ_i are algebraic generators, therefore Γ_i satisfies the relation$\Gamma_i\Gamma_j\cdots \neq \Gamma_t$, where $\Gamma_i, \Gamma_j, \ldots, \Gamma_t \in C\ell_m(K)$. On the other hand, the element of the decomposed Clifford algebra $\Gamma_i^{(\pm)}$ does not satisfy the relation $\Gamma_i^{(\pm)}\Gamma_j^{(\pm)}\cdots \neq \Gamma_t^{(\pm)}$, where $\Gamma_i^{(\pm)}, \Gamma_j^{(\pm)}, \ldots, \Gamma_t^{(\pm)} \in C\ell_m^{(\pm)}(K)$.

We can construct the $4n$-dimensional ASD tensor $\Sigma_{\mu\nu}^{(\pm)}$ form the $(4n-1)$-dimensional Clifford algebra $C\ell_{4n-1}(K)$. Here the $4n$-dimensional "ASD tensor" means that the tensor satisfies the ASD relation in $4n$ dimensions. We define the $4n$-dimensional basis e_μ by

$$e_\mu = \delta_{\mu 4n}1 + \delta_{\mu i}\Gamma_i^{(-)}, \quad e_\mu^\dagger = \delta_{\mu 4n}1 + \delta_{\mu i}\Gamma_i^{(+)}, \tag{2.167}$$

where the indices $\mu, \nu, \ldots$ run from 1 to $4n$. Using this basis, we define the $4n$-dimensional ASD tensor by

$$\Sigma_{\mu\nu}^{(+)} = e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu, \quad \Sigma_{\mu\nu}^{(-)} = e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger. \tag{2.168}$$

We can confirm that $\Sigma_{\mu\nu}^{(\pm)}$ satisfies the $4n$-dimensional ASD relation:

$$\Sigma_{[a_1a_2}^{(\pm)}\ldots\Sigma_{a_{2n-1}a_{2n}]}^{(\pm)} = \pm\frac{1}{2n!}\varepsilon_{a_1a_2\ldots a_{2n}b_1b_2\ldots b_{2n}}\Sigma_{b_1b_2}^{(\pm)}\ldots\Sigma_{b_{2n-1}b_{2n}}^{(\pm)} \tag{2.169}$$

where $\Sigma_{\mu\nu}^{(+)}$ satisfies the self-dual equation and $\Sigma_{\mu\nu}^{(-)}$ satisfies the anti-self-dual equation respectively. We already showed that the ASD basis which is constructed by above method satisfies the ASD relation at theorem 2.4.1 on p.33.

In order to discuss the $4n$-dimensional ADHM construction in more detail, we need the explicit representations of the ASD basis namely the matrix representations of the $(4n-1)$-dimensional Clifford algebra $C\ell_{4n-1}(K)$. Fortunately, we already know that the real/complex Clifford algebras have the isomorphism with the matrix rings. For the Bott periodicity theorem, we summarize the $(4n-1)$-dimensions results into Table 2.3. This result is strong backbone to construct the matrix representations of ASD basis. Although the higher-dimensional real Clifford algebra case is difficult in mathematically, we can represent the complex Clifford algebra by the matrix in any $4n-1$ dimensions. Therefore we construct the four- and eight-dimensional ASD basis explicitly as follows. Furthermore we show the matrix representation of the $(4n-1)$-dimenaional complex Clifford algebra.

[5] Strictly speaking, the case of the complex Clifford algebra is able to $m = 2n-1$.

We note that the representation of $\Gamma_i^{(\pm)}$ is not uniqueness because there is the following transformation that holds the relation $\{\Gamma_i^{(\pm)}, \Gamma_j^{(\pm)}\} = -2\delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}$:

$$\Gamma_\mu^{(\pm)} \mapsto \tilde{\Gamma}_i^{(\pm)} = M\Gamma_i^{(\pm)}M^{-1}, \tag{2.170}$$

where $M \in \mathrm{U}(2^{2n-1})$. Therefore we find that the ASD basis e_μ has the following freedom for the representations:

$$e_\mu \mapsto \tilde{e}_\mu = Me_\mu M^{-1}, \quad e_\mu^\dagger \mapsto \tilde{e}_\mu^\dagger = Me_\mu^\dagger M^{-1}. \tag{2.171}$$

We use the tensor product of the following 2×2 matrices. The complex Clifford algebra $C\ell_m(\mathbb{C})$ is constructed by the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \mathbf{1}_2. \tag{2.172}$$

On the other hand, the real Clifford algebras $C\ell_m(\mathbb{R})$ are constructed by the following matrices [39]:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_0 = \mathbf{1}_2. \tag{2.173}$$

For simplicity, we omit the tensor (Kronecker) product symbol $\otimes$ in the following discussions. For example, σ_{ij} means $\sigma_i \otimes \sigma_j$.

The complex basis in four dimensions

We construct the four-dimensional ASD tensor from the three-dimensional Clifford algebra. The matrix representation of the three-dimensional complex Clifford algebra $C\ell_3(\mathbb{C})$ is given by

$$\Gamma_1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \ \Gamma_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \ \Gamma_3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}. \tag{2.174}$$

The chiral element ω and the projection operators $P_\pm$ are

$$\omega = \Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \qquad P_+ = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & 0 \end{pmatrix}, \ P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}. \tag{2.175}$$

Using these matrices, we obtain

$$\Gamma_i^{(\pm)} = \pm i\sigma_i, \tag{2.176}$$

where $i = 1, 2, 3$. Therefore we obtain the four-dimensional ASD complex basis:

$$e_\mu = \delta_{\mu 4}\mathbf{1}_2 - i\delta_{\mu i}\sigma_i, \ \ e_\mu^\dagger = \delta_{\mu 4}\mathbf{1}_2 + i\delta_{\mu i}\sigma_i. \tag{2.177}$$

This basis is just the quaternion basis which is used in the four-dimensional ADHM construction.

The real basis in four dimensions

For Table 2.3, the three-dimensional real Clifford algebra $C\ell_3(\mathbb{R})$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$. However we use real matrix representation to implement the orthogonal gauge group. The real matrix representation of $C\ell_3(\mathbb{R})$ is given by

$$\Gamma_1 = \begin{pmatrix} \tau_{12} & 0 \\ 0 & -\tau_{12} \end{pmatrix}, \ \Gamma_2 = \begin{pmatrix} \tau_{20} & 0 \\ 0 & -\tau_{20} \end{pmatrix}, \ \Gamma_3 = \begin{pmatrix} \tau_{32} & 0 \\ 0 & -\tau_{32} \end{pmatrix}. \tag{2.178}$$

The chiral element ω and the projection operators $P_\pm$ are

$$\omega = \Gamma_1\Gamma_2\Gamma_3 = \begin{pmatrix} -\mathbf{1}_4 & 0 \\ 0 & \mathbf{1}_4 \end{pmatrix}, \qquad P_+ = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_4 \end{pmatrix}, \ P_- = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.179}$$

Therefore $\Gamma_i^{(\pm)}$ are

$$\Gamma_1^{(\pm)} = \mp\tau_{12}, \quad \Gamma_2^{(\pm)} = \mp\tau_{20}, \quad \Gamma_3^{(\pm)} = \mp\tau_{32}, \tag{2.180}$$

and we obtain the four-dimensional ASD tensor by using (2.167) and (2.168).

$$x^\mu e_\mu = \begin{pmatrix} x^4 & -x^3 & -x^2 & -x^1 \\ x^3 & x^4 & x^1 & -x^2 \\ x^2 & -x^1 & x^4 & x^3 \\ x^1 & x^2 & -x^3 & x^4 \end{pmatrix}. \tag{2.181}$$

If this real basis is used in the four-dimensional ADHM construction, the gauge group becomes $G = \mathrm{O}(N)$.

The complex basis in eight dimensions

Type I The matrix representation of $C\ell_7(\mathbb{C})$ is given by

$$\begin{aligned}&\Gamma_1=\begin{pmatrix} i\sigma_{133} & 0\\ 0 & -i\sigma_{133}\end{pmatrix},\ \Gamma_2=\begin{pmatrix} i\sigma_{233} & 0\\ 0 & -i\sigma_{233}\end{pmatrix},\ \Gamma_3=\begin{pmatrix} i\sigma_{013} & 0\\ 0 & -i\sigma_{013}\end{pmatrix},\\ &\Gamma_4=\begin{pmatrix} i\sigma_{023} & 0\\ 0 & -i\sigma_{023}\end{pmatrix},\ \Gamma_5=\begin{pmatrix} i\sigma_{001} & 0\\ 0 & -i\sigma_{001}\end{pmatrix},\ \Gamma_6=\begin{pmatrix} i\sigma_{002} & 0\\ 0 & -i\sigma_{002}\end{pmatrix},\ \Gamma_7=\begin{pmatrix} i\sigma_{333} & 0\\ 0 & -i\sigma_{333}\end{pmatrix}.\end{aligned} \tag{2.182}$$

Using (2.164b), the chiral element ω is given by

$$\omega=(-1)\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7=\begin{pmatrix}\mathbf{1}_8 & 0\\ 0 & -\mathbf{1}_8\end{pmatrix}. \tag{2.183}$$

The projection operators $P_\pm$ are

$$P_+=\begin{pmatrix}\mathbf{1}_8 & 0\\ 0 & 0\end{pmatrix},\quad P_-=\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}_8\end{pmatrix}. \tag{2.184}$$

Therefore we obtain

$$\begin{aligned}&\Gamma_1^{(\pm)}=\pm i\sigma_{133},\ \Gamma_2^{(\pm)}=\pm i\sigma_{233},\ \Gamma_3^{(\pm)}=\pm i\sigma_{013},\\ &\Gamma_4^{(\pm)}=\pm i\sigma_{023},\ \Gamma_5^{(\pm)}=\pm i\sigma_{001},\ \Gamma_6^{(\pm)}=\pm i\sigma_{002},\ \Gamma_7^{(\pm)}=\pm i\sigma_{333}.\end{aligned} \tag{2.185}$$

We now define $\tilde z_1 := x^2+ix^1,\ \tilde z_2 := x^4+ix^3,\ \tilde z_3 := x^6+ix^5,\ \tilde z_4 := x^8+ix^7$ and these complex conjugate denote $\tilde z_i^\dagger$.

$$x^\mu e_\mu=\begin{pmatrix}
x^8-ix^7 & -x^6-ix^5 & -x^4-ix^3 & 0 & -x^2-ix^1 & 0 & 0 & 0\\
x^6-ix^5 & x^8+ix^7 & 0 & x^4+ix^3 & 0 & x^2+ix^1 & 0 & 0\\
x^4-ix^3 & 0 & x^8+ix^7 & -x^6-ix_5 & 0 & 0 & x^2+ix^1 & 0\\
0 & -x^4+ix^3 & x^6-ix^5 & x^8-ix^7 & 0 & 0 & 0 & -x^2-ix^1\\
x^2-ix^1 & 0 & 0 & 0 & x^8+ix^7 & -x^6-ix^5 & -x^4-ix^3 & 0\\
0 & -x^2+ix^1 & 0 & 0 & x^6-ix^5 & x^8-ix^7 & 0 & x^4+ix^3\\
0 & 0 & -x^2+ix^1 & 0 & x^4-ix^3 & 0 & x^8-ix^7 & -x^6-ix^5\\
0 & 0 & 0 & x^2-ix^1 & 0 & -x^4+ix^3 & x^6-ix^5 & x^8+ix^7
\end{pmatrix} \tag{2.186a}$$

$$=\begin{pmatrix}
\tilde z_4^\dagger & -\tilde z_3 & -\tilde z_2 & 0 & -\tilde z_1 & 0 & 0 & 0\\
\tilde z_3^\dagger & \tilde z_4 & 0 & \tilde z_2 & 0 & \tilde z_1 & 0 & 0\\
\tilde z_2^\dagger & 0 & \tilde z_4 & -\tilde z_3 & 0 & 0 & \tilde z_1 & 0\\
0 & -\tilde z_2^\dagger & \tilde z_3^\dagger & \tilde z_4^\dagger & 0 & 0 & 0 & -\tilde z_1\\
\tilde z_1^\dagger & 0 & 0 & 0 & \tilde z_4 & -\tilde z_3 & -\tilde z_2 & 0\\
0 & -\tilde z_1^\dagger & 0 & 0 & \tilde z_3^\dagger & \tilde z_4^\dagger & 0 & \tilde z_2\\
0 & 0 & -\tilde z_1^\dagger & 0 & \tilde z_2^\dagger & 0 & \tilde z_4^\dagger & -\tilde z_3\\
0 & 0 & 0 & \tilde z_1^\dagger & 0 & -\tilde z_2^\dagger & \tilde z_3^\dagger & \tilde z_4
\end{pmatrix}=\begin{pmatrix}\tilde B+\tilde C^\dagger & \tilde A\\ -\tilde A^\dagger & \tilde B+\tilde C\end{pmatrix}, \tag{2.186b}$$

where $\tilde A, \tilde B, \tilde C$ are the 4 complex matrices which are defined by

$$\tilde A := \begin{pmatrix}-\tilde z_1 & 0 & 0 & 0\\ 0 & \tilde z_1 & 0 & 0\\ 0 & 0 & \tilde z_1 & 0\\ 0 & 0 & 0 & -\tilde z_1\end{pmatrix},\qquad \tilde B := \begin{pmatrix}0 & -\tilde z_3 & -\tilde z_2 & 0\\ \tilde z_3^\dagger & 0 & 0 & \tilde z_2\\ \tilde z_2^\dagger & 0 & 0 & -\tilde z_3\\ 0 & -\tilde z_2^\dagger & \tilde z_3^\dagger & 0\end{pmatrix},\qquad \tilde C := \begin{pmatrix}\tilde z_4 & 0 & 0 & 0\\ 0 & \tilde z_4^\dagger & 0 & 0\\ 0 & 0 & \tilde z_4^\dagger & 0\\ 0 & 0 & 0 & \tilde z_4\end{pmatrix}, \tag{2.187}$$

Type II Of course, we can take another matrix representation:

$$\begin{aligned}&\Gamma_1=\begin{pmatrix} i\sigma_{112} & 0\\ 0 & -i\sigma_{112}\end{pmatrix},\ \Gamma_2=\begin{pmatrix} i\sigma_{120} & 0\\ 0 & -i\sigma_{120}\end{pmatrix},\ \Gamma_3=\begin{pmatrix} -i\sigma_{132} & 0\\ 0 & i\sigma_{132}\end{pmatrix},\\ &\Gamma_4=\begin{pmatrix} -i\sigma_{221} & 0\\ 0 & i\sigma_{221}\end{pmatrix},\ \Gamma_5=\begin{pmatrix} i\sigma_{223} & 0\\ 0 & -i\sigma_{223}\end{pmatrix},\ \Gamma_6=\begin{pmatrix} -i\sigma_{202} & 0\\ 0 & i\sigma_{202}\end{pmatrix},\ \Gamma_7=\begin{pmatrix} i\sigma_{300} & 0\\ 0 & -i\sigma_{300}\end{pmatrix}.\end{aligned} \tag{2.188}$$

In this case, the chiral element ω becomes same as type I thus the projection operators becomes same as type I also. Therefore $\Gamma_i^{(\pm)}$ are

$$\begin{aligned}&\Gamma_1^{(\pm)} = \pm i\sigma_{112},\ \Gamma_2^{(\pm)} = \pm i\sigma_{120},\ \Gamma_3^{(\pm)} = \mp i\sigma_{132},\\&\Gamma_4^{(\pm)} = \mp i\sigma_{221},\ \Gamma_5^{(\pm)} = \pm i\sigma_{223},\ \Gamma_6^{(\pm)} = \mp i\sigma_{202},\quad \Gamma_7^{(\pm)} = \pm i\sigma_{300},\end{aligned} \tag{2.189}$$

The basis (2.189) is used to construct the Grossman's one-instantons [31].

We now define $z_1 := x^1 + ix^4,\ z_2 := x^2 + ix^5,\ z_3 := x^3 + ix^6,\ z_4 := x^8 + ix^7$ and these complex conjugate denote $z_i^\dagger$.

$$x^\mu e_\mu = \begin{pmatrix} x^8 - ix^7 & 0 & 0 & 0 & 0 & x^3 - ix^6 & -x^2 + ix^5 & -x^1 - ix^4 \\ 0 & x^8 - ix^7 & 0 & 0 & -x^3 + ix^6 & 0 & x^1 - ix^4 & -x^2 - ix^5 \\ 0 & 0 & x^8 - ix^7 & 0 & x^2 - ix^5 & -x^1 + ix^4 & 0 & -x^3 - ix^6 \\ 0 & 0 & 0 & x^8 - ix^7 & x^1 + ix^4 & x^2 + ix^5 & x^3 + ix^6 & 0 \\ 0 & x^3 + ix^6 & -x^2 - ix^5 & -x^1 + ix^4 & x^8 + ix^7 & 0 & 0 & 0 \\ -x^3 - ix^6 & 0 & x^1 + ix^4 & -x^2 + ix^5 & 0 & x^8 + ix^7 & 0 & 0 \\ x^2 + ix^5 & -x^1 - ix^4 & 0 & -x^3 + ix^6 & 0 & 0 & x^8 + ix^7 & 0 \\ x^1 - ix^4 & x^2 - ix^5 & x^3 - ix^6 & 0 & 0 & 0 & 0 & x^8 + ix^7 \end{pmatrix} \tag{2.190a}$$

$$= \begin{pmatrix} z_4^\dagger & 0 & 0 & 0 & 0 & z_3^\dagger & -z_2^\dagger & -z_1 \\ 0 & z_4^\dagger & 0 & 0 & -z_3^\dagger & 0 & z_1^\dagger & -z_2 \\ 0 & 0 & z_4^\dagger & 0 & z_2^\dagger & -z_1^\dagger & 0 & -z_3 \\ 0 & 0 & 0 & z_4^\dagger & z_1 & z_2 & z_3 & 0 \\ 0 & z_3 & -z_2 & -z_1^\dagger & z_4 & 0 & 0 & 0 \\ -z_3 & 0 & z_1 & -z_2^\dagger & 0 & z_4 & 0 & 0 \\ z_2 & -z_1 & 0 & -z_3^\dagger & 0 & 0 & z_4 & 0 \\ z_1^\dagger & z_2^\dagger & z_3^\dagger & 0 & 0 & 0 & 0 & z_4 \end{pmatrix} = \begin{pmatrix} B^\dagger & A \\ -A^\dagger & B \end{pmatrix}, \tag{2.190b}$$

where A, B are the 4 complex matrices which are defined by

$$A := \begin{pmatrix} 0 & z_3^\dagger & -z_2^\dagger & -z_1 \\ -z_3^\dagger & 0 & z_1^\dagger & -z_2 \\ z_2^\dagger & -z_1^\dagger & 0 & -z_3 \\ z_1 & z_2 & z_3 & 0 \end{pmatrix}, \qquad B := z_4 \mathbf{1}_4. \tag{2.191}$$

The real basis in eight dimensions

The matrix representation of $C\ell_7(\mathbb{R})$ is given by

$$\begin{aligned}&\Gamma_1 = \begin{pmatrix} \tau_{222} & 0 \\ 0 & -\tau_{222} \end{pmatrix},\ \Gamma_2 = \begin{pmatrix} \tau_{012} & 0 \\ 0 & -\tau_{012} \end{pmatrix},\ \Gamma_3 = \begin{pmatrix} \tau_{201} & 0 \\ 0 & -\tau_{201} \end{pmatrix},\\&\Gamma_4 = \begin{pmatrix} \tau_{032} & 0 \\ 0 & -\tau_{032} \end{pmatrix},\ \Gamma_5 = \begin{pmatrix} \tau_{120} & 0 \\ 0 & -\tau_{120} \end{pmatrix},\ \Gamma_6 = \begin{pmatrix} \tau_{320} & 0 \\ 0 & -\tau_{320} \end{pmatrix},\ \Gamma_7 = \begin{pmatrix} \tau_{203} & 0 \\ 0 & -\tau_{203} \end{pmatrix}.\end{aligned} \tag{2.192}$$

Using (2.164a), the chiral element ω is given by

$$\omega = (-1)\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_7 = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & -\mathbf{1}_8 \end{pmatrix}. \tag{2.193}$$

The projection operators $P_\pm$ are

$$P_+ = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & 0 \end{pmatrix},\ P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_8 \end{pmatrix}. \tag{2.194}$$

Therefore we obtain

$$\begin{aligned}&\Gamma_1^{(\pm)} = \pm\tau_{222},\ \Gamma_2^{(\pm)} = \pm\tau_{012},\ \Gamma_3^{(\pm)} = \pm\tau_{201},\\&\Gamma_4^{(\pm)} = \pm\tau_{032},\ \Gamma_5^{(\pm)} = \pm\tau_{120},\ \Gamma_6^{(\pm)} = \pm\tau_{320},\ \Gamma_7^{(\pm)} = \pm\tau_{203}.\end{aligned} \tag{2.195}$$

$$x^\mu e_\mu = \begin{pmatrix} x^8 & x^4 & x^6 & x^2 & x^7 & x^3 & x^5 & x^1 \\ -x^4 & x^8 & -x^2 & x^6 & x^3 & -x^7 & -x^1 & x^5 \\ -x^6 & x^2 & x^8 & -x^4 & -x^5 & -x^1 & x^7 & x^3 \\ -x^2 & -x^6 & x^4 & x^8 & x^1 & -x^5 & x^3 & -x^7 \\ -x^7 & -x^3 & x^5 & -x^1 & x^8 & x^4 & -x^6 & x^2 \\ -x^3 & x^7 & x^1 & x^5 & -x^4 & x^8 & -x^2 & -x^6 \\ -x^5 & x^1 & -x^7 & -x^3 & x^6 & x^2 & x^8 & -x^4 \\ -x^1 & -x^5 & -x^3 & x^7 & -x^2 & x^6 & x^4 & x^8 \end{pmatrix} \tag{2.196}$$

The complex basis in $4n$ dimensions

There are systematic construction of the matrix representation of the complex Clifford algebras [40]. Using this constrution, we obtain the $(4n-1)$-dimensional decomposed Clifford algebra:

$$\begin{aligned} \Gamma_1^{(\pm)} &= \pm i\sigma_1^{(1)} \otimes \sigma_3^{(2)} \otimes \cdots \otimes \sigma_3^{(2n-1)}, & \Gamma_2^{(\pm)} &= \pm i\sigma_2^{(1)} \otimes \sigma_3^{(2)} \otimes \cdots \otimes \sigma_3^{(2n-1)}, \\ \Gamma_3^{(\pm)} &= \pm i\sigma_0^{(1)} \otimes \sigma_1^{(2)} \otimes \cdots \otimes \sigma_3^{(2n-1)}, & \Gamma_4^{(\pm)} &= \pm i\sigma_0^{(1)} \otimes \sigma_2^{(2)} \otimes \cdots \otimes \sigma_3^{(2n-1)}, \\ &\vdots & &\vdots \\ \Gamma_{4n-3}^{(\pm)} &= \pm i\sigma_0^{(1)} \otimes \sigma_0^{(2)} \otimes \cdots \otimes \sigma_1^{(2n-1)}, & \Gamma_{4n-2}^{(\pm)} &= \pm i\sigma_0^{(1)} \otimes \sigma_0^{(2)} \otimes \cdots \otimes \sigma_2^{(2n-1)}, \\ \Gamma_{4n-1}^{(\pm)} &= \pm i\sigma_3^{(1)} \otimes \sigma_3^{(2)} \otimes \cdots \otimes \sigma_3^{(2n-1)}, \end{aligned} \tag{2.197}$$

where σ_i are the Pauli matrices and $\sigma_0 = \mathbf{1}_2$. These matrices are multiplied imaginary with the matrix as the Brauer-Weyl matrices. The Brauer-Weyl matrix is often used as the higher dimensional gamma matrix in Physics.

2.6 The higher dimensional ADHM equation

In four dimension, we could rewrite the ADHM constraint with canonical form to more easily formal which usually called the ADHM equation (1.52). Similarly, for previous subsection 2.2, we have found that the higher dimensional ADHM constraints with canonical form are able to rewritten as the ADHM equations (2.42), (2.43) and (2.44). Thus we have the following question: What is character of the ADHM equations in higher dimensions? Let us examine the higher dimensional ADHM equation in more detail to solve this question.

In four dimensions, the property of the ASD basis as $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$ $(i, j, k = 1, 2, 3)$ is essential to lead the ADHM equation. This property means that the ASD basis, which is the quaternion or the Pauli matrices, is closure under the multiplication. Unfortunately the higher dimensional ASD basis which is obtained from the Clifford algebra is not closure under the multiplication. It is not straightforward that we algebraically lead the higher dimensional equation by the method as similar as the four dimensional one, but is benefitical to discuss the algebraic property of the ADHM equation in higher dimensions. We first consider the ADHM equation that is associated with the first ADHM constraint (2.42). $T^\dagger T$ is expanded as

$$T^\dagger T = e_\mu^\dagger e_\nu \otimes T^\mu T^\nu = \left(\delta_{\mu\nu}\mathbf{1}_{2^{2n-1}} + \frac{1}{2}\Sigma_{\mu\nu}^{(+)}\right) \otimes T^\mu T^\nu = \mathbf{1}_{2^{2n-1}} \otimes T^2 + \frac{1}{2}\Sigma_{\mu\nu}^{(+)} \otimes T^\mu T^\nu. \tag{2.198}$$

For (2.105) and (2.106a),

$$\begin{aligned} \Sigma_{\mu\nu}^{(+)} &= e_\mu^\dagger e_\nu - e_\nu^\dagger e_\mu \\ &= 2\left(e_\mu^\dagger e_\nu - \delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}\right) \\ &= 2\left\{(\delta_{\mu\#}\mathbf{1}_{2^{2n-1}} + \delta_{\mu i}\gamma_i)(\delta_{\nu\#}\mathbf{1}_{2^{2n-1}} - \delta_{\nu j}\gamma_j) - \delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}\right\} \\ &= 2(\delta_{\mu\#}\delta_{\nu\#}\mathbf{1}_{2^{2n-1}} - \delta_{\mu\#}\delta_{\nu j}\gamma_j + \delta_{\nu\#}\delta_{\mu i}\gamma_i - \delta_{\mu i}\delta_{\nu j}\gamma_i\gamma_j - \delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}). \end{aligned} \tag{2.199}$$

In the case of four dimensions, the fourth term in above equation becomes $\gamma_i\gamma_j = -\sigma_i\sigma_j = -i\varepsilon_{ijk}\sigma_k = -\varepsilon_{ijk}\gamma_k$ $(i, j, k = 1, 2, 3)$, thus we algebraic lead the ADHM equations easily. On the other hand, in higher dimensions, the fourth term is not proportional to γ_i, i.e. the ASD basis is not closure , thus we can not lead a practical algebraic form of the ADHM equation which is similar to the four dimensions one (1.52).

Now we introduce the basis X_α which is defined by the multiplication of γ_i: $\{X_a\} \coloneqq \{\gamma_i\gamma_j\}$ $(i \neq j)$. Moreover we define the set of basis Y_β by the sum sets of γ_i and X_a:

$$\{Y_\beta\} \coloneqq \{\gamma_i\} \oplus \{X_\alpha\} = \{\gamma_i\} \oplus \{\gamma_i\gamma_j\}. \quad (i \neq j) \tag{2.200}$$

Expand the ASD tensor $\Sigma^{(+)}_{\mu\nu}$ with using X_α, Y_β:

$$\begin{aligned}\Sigma^{(+)}_{\mu\nu} &= 2(\delta_{\mu\#}\delta_{\nu\#}\mathbf{1}_{2^{2n-1}} - \delta_{\mu\#}\delta_{\nu j}\gamma_j + \delta_{\nu\#}\delta_{\mu i}\gamma_i - \delta_{\mu i}\delta_{\nu j}\gamma_i\gamma_j - \delta_{\mu\nu}\mathbf{1}_{2^{2n-1}}) \\ &= 2\left(-\delta_{\mu\#}\delta_{\nu j}\gamma_j + \delta_{\nu\#}\delta_{\mu i}\gamma_i + \mathcal{F}^{\alpha}_{\mu\nu}X_\alpha\right) \qquad \because \gamma_i\gamma_i = -\mathbf{1}_{2^{2n-1}} \text{ (not summed)} \\ &=: 2\Sigma^{\beta(+)}_{\mu\nu}Y_\beta,\end{aligned} \tag{2.201}$$

where $\mathcal{F}^{\alpha}_{\mu\nu}$ is a third-order tensor that is satisfy $\mathcal{F}^{\alpha}_{\mu\nu}X_\alpha = -\delta_{\mu i}\delta_{\nu j}\gamma_i\gamma_j$ $(\mu \neq \nu)$. Thus (2.198) becomes

$$T^\dagger T = \mathbf{1}_{2^{2n-1}} \otimes T^2 + Y_\beta \otimes \Sigma^{\beta(+)}_{\mu\nu}T^\mu T^\nu. \tag{2.202}$$

Similarly, let us consider that $S^\dagger S$ term is expanded by suitable basis. Z_γ denotes the new basis which are satisfy as

$$S^\dagger S = \mathbf{1}_{2^{2n-1}} \otimes \mathcal{S}_{(1)} + Y_\beta \otimes \mathcal{S}^\beta_{(2)} + Z_\gamma \otimes \mathcal{S}^\gamma_{(3)}. \tag{2.203}$$

Here $\mathcal{S}_{(1)}, \mathcal{S}_{(2)}, \mathcal{S}_{(3)}$ are $k \times k$ matrices. By definition of S, for all $S^\dagger S$ is able to written as $^\exists\mathcal{I}^a \in \mathrm{U}(k)$ *s.t.* $S^\dagger S = A_a \otimes \mathcal{I}^a$ where A_a are the group elements of $\mathrm{U}(2^{2n-1})$ and $a = 1, 2, \ldots, 2^{2(2n-1)}$. If four dimensional case ($n = 1$) then A_a are the group elements of U(2), thus A_a are just the four-dimensional ASD bais: $A_\mu = e_\mu$ $(\mu = 1, 2, 3, 4)$. This fact means that the ASD basis e_μ spans $S^\dagger S$, namely, in four dimensions , we do not have to introduce the new basis Z_γ in (2.203). On the other hand, in higher dimensions ($n \geq 2$), the ASD basis e_μ can not spans $S^\dagger S$. This fact is easily shown as follows. If the ASD bais e_μ (and these maltiply $e_\mu e_\nu$) span $S^\dagger S$ then the elements number of set $\{\mathbf{1}_{2^{2n-1}}\} \oplus \{Y_\beta\}$ equal as (or more than) the elements number of basis set $\mathrm{U}(2^{2n-1})$. However the elements number of set $\{\mathbf{1}_{2^{2n-1}}\} \oplus \{Y_\beta\}$ becomes $1 + (4n-1) + (4n-1)\cdot(4n-2)/2 = 8n^2 - 2n + 1$ and the elements number of basis set $\mathrm{U}(2^{2n-1})$ is $2^{2(2n-1)}$, $2^{2(2n-1)} > 8n^2 - 2n + 1$ in higher dimensions ($n \geq 2$). Therefore the ASD basis does not span $S^\dagger S$ in higher dimensions ($n \geq 2$). For this reason, we have been needing the new basis Z_γ to expand $S^\dagger S$ in higher dimensions.

Here we expand the first ADHM constraint with using (2.202) and (2.203),

$$\begin{aligned}T^\dagger T + S^\dagger S = \mathbf{1}_{2^{2n-1}} \otimes E^{(1)}_k &\iff \mathbf{1}_{2^{2n-1}} \otimes T^2 + Y_\beta \otimes \Sigma^{\beta(+)}_{\mu\nu}T^\mu T^\nu + \mathbf{1}_{2^{2n-1}} \otimes \mathcal{S}_{(1)} + Y_\beta \otimes \mathcal{S}^\beta_{(2)} + Z_\gamma \otimes \mathcal{S}^\gamma_{(3)} = \mathbf{1}_{2^{2n-1}} \otimes E^{(1)}_k, \\ &\iff \mathbf{1}_{2^{2n-1}} \otimes \left(T^2 + \mathcal{S}_{(1)}\right) + Y_\beta \otimes \left(\Sigma^{\beta(+)}_{\mu\nu}T^\mu T^\nu + \mathcal{S}^\beta_{(2)}\right) + Z_\gamma \otimes \mathcal{S}^\gamma_{(3)} = \mathbf{1}_{2^{2n-1}} \otimes E^{(1)}_k.\end{aligned} \tag{2.204}$$

thus we obtain

$$\Sigma^{\beta(+)}_{\mu\nu}T^\mu T^\nu + \mathcal{S}^\beta_{(2)} = 0, \tag{2.205a}$$

$$\mathcal{S}^\gamma_{(3)} = 0. \tag{2.205b}$$

The first equation is a higher-dimensionalization of the four dimensional ADHM equation which is combined with the T term and the S term. On the other hand, the second equation is a new type equation with only the S term. Next, as one example, let us lead the eight dimensional ADHM equation in explicitly.

2.6.1 An eight-dimensional U(8) ADHM equaiton

In this subsection, we will lead an explicitly eight-dimensional ADHM equations with U(8) gauge group. Here we use the basis (2.189).

Let us lead the explicit form of the ADHM equations which are associated with the first ADHM constraint (2.42), we call these equations as “the first ADHM equations”. Now we recall $T = e_\mu \otimes T^\mu$ in (2.41), then the Weyl operator without x terms is rewritten to

$$
\begin{aligned}
\tilde{\Delta} &= \begin{pmatrix} S_{[8]\times[8k]} \\ e_{\mu\ [8]} \otimes T^{\mu}_{[k]} \end{pmatrix}_{[8+8k]\times[8k]} \\
&= \begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 \\
T^8 - iT^7 & 0 & 0 & 0 & 0 & T^3 - iT^6 & -T^2 + iT^5 & -T^1 - iT^4 \\
0 & T^8 - iT^7 & 0 & 0 & -T^3 + iT^6 & 0 & T^1 - iT^4 & -T^2 - iT^5 \\
0 & 0 & T^8 - iT^7 & 0 & T^2 - iT^5 & -T^1 + iT^4 & 0 & -T^3 - iT^6 \\
0 & 0 & 0 & T^8 - iT^7 & T^1 + iT^4 & T^2 + iT^5 & T^3 + iT^6 & 0 \\
0 & T^3 + iT^6 & -T^2 - iT^5 & -T^1 + iT^4 & T^8 + iT^7 & 0 & 0 & 0 \\
-T^3 - iT^6 & 0 & T^1 + iT^4 & -T^2 + iT^5 & 0 & T^8 + iT^7 & 0 & 0 \\
T^2 + iT^5 & -T^1 - iT^4 & 0 & -T^3 + iT^6 & 0 & 0 & T^8 + iT^7 & 0 \\
T^1 - iT^4 & T^2 - iT^5 & T^3 - iT^6 & 0 & 0 & 0 & 0 & T^8 + iT^7
\end{pmatrix} \\
&= \begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 \\
L_4^\dagger & 0 & 0 & 0 & 0 & L_3^\dagger & -L_2^\dagger & -L_1 \\
0 & L_4^\dagger & 0 & 0 & -L_3^\dagger & 0 & L_1^\dagger & -L_2 \\
0 & 0 & L_4^\dagger & 0 & L_2^\dagger & -L_1^\dagger & 0 & -L_3 \\
0 & 0 & 0 & L_4^\dagger & L_1 & L_2 & L_3 & 0 \\
0 & L_3 & -L_2 & -L_1^\dagger & L_4 & 0 & 0 & 0 \\
-L_3 & 0 & L_1 & -L_2^\dagger & 0 & L_4 & 0 & 0 \\
L_2 & -L_1 & 0 & -L_3^\dagger & 0 & 0 & L_4 & 0 \\
L_1^\dagger & L_2^\dagger & L_3^\dagger & 0 & 0 & 0 & 0 & L_4
\end{pmatrix} =: \begin{pmatrix}
\Sigma_{1\ [8]\times[4k]} & \Sigma_{2\ [8]\times[4k]} \\
\Xi^\dagger_{2\ [4k]} & \Xi_{1\ [4k]} \\
-\Xi^\dagger_{1\ [4k]} & \Xi_{2\ [4k]}
\end{pmatrix},
\end{aligned}
\tag{2.206}
$$

where the matrix subscript $[a] \times [b]$ means the matrix size, and we have defined as $L_1 := T^1 + iT^4,\ L_2 := T^2 + iT^5,\ L_3 := T^3 + iT^6,\ L_4 := T^8 + iT^7$. Now we recall that T^μ is Hermite, thus $L_1^\dagger = T^1 - iT^4,\ L_2^\dagger := T^2 - iT^5$, etc. We have defined by $8 \times 4k$ matrices: $\Sigma_{1,2}$ and $4k \times 4k$ matrices $\Xi_{1,2}$ as follows:

$$
\begin{aligned}
&\Sigma_1 := \begin{pmatrix} S_1 & S_2 & S_3 & S_4 \end{pmatrix}, \qquad &&\Sigma_2 := \begin{pmatrix} S_5 & S_6 & S_7 & S_8 \end{pmatrix}, \\
&\Xi_1 := \begin{pmatrix}
0 & L_3^\dagger & -L_2^\dagger & -L_1 \\
-L_3^\dagger & 0 & L_1^\dagger & -L_2 \\
L_2^\dagger & -L_1^\dagger & 0 & -L_3 \\
L_1 & L_2 & L_3 & 0
\end{pmatrix}, \qquad &&\Xi_2 := \begin{pmatrix}
L_4 & 0 & 0 & 0 \\
0 & L_4 & 0 & 0 \\
0 & 0 & L_4 & 0 \\
0 & 0 & 0 & L_4
\end{pmatrix} = \mathbf{1}_4 \otimes L_4
\end{aligned}
\tag{2.207}
$$

Similarly, the conjugate of the Weyl operator without x terms is

$$
\begin{aligned}
\tilde{\Delta}^\dagger &= \begin{pmatrix} S^\dagger & e_\mu^\dagger \otimes T^\mu \end{pmatrix} \\
&= \begin{pmatrix}
S_1^\dagger & L_4 & 0 & 0 & 0 & 0 & -L_3^\dagger & L_2^\dagger & L_1 \\
S_2^\dagger & 0 & L_4 & 0 & 0 & L_3^\dagger & 0 & -L_1^\dagger & L_2 \\
S_3^\dagger & 0 & 0 & L_4 & 0 & -L_2^\dagger & L_1^\dagger & 0 & L_3 \\
S_4^\dagger & 0 & 0 & 0 & L_4 & -L_1 & -L_2 & -L_3 & 0 \\
S_5^\dagger & 0 & -L_3 & L_2 & L_1^\dagger & L_4^\dagger & 0 & 0 & 0 \\
S_6^\dagger & L_3 & 0 & -L_1 & +L_2^\dagger & 0 & L_4^\dagger & 0 & 0 \\
S_7^\dagger & -L_2 & L_1 & 0 & L_3^\dagger & 0 & 0 & L_4^\dagger & 0 \\
S_8^\dagger & -L_1^\dagger & -L_2^\dagger & -L_3^\dagger & 0 & 0 & 0 & 0 & L_4^\dagger
\end{pmatrix} \\
&= \begin{pmatrix}
\Sigma_1^\dagger & \Xi_2 & -\Xi_1 \\
\Sigma_2^\dagger & \Xi_1^\dagger & \Xi_2^\dagger
\end{pmatrix}.
\end{aligned}
\tag{2.208}
$$

For (2.206) and (2.208), the l.h.s. of the first ADHM equations (2.42) become

$$
S^\dagger S + T^\dagger T = \tilde{\Delta}^\dagger \tilde{\Delta} = \begin{pmatrix}
\Sigma_1^\dagger & \Xi_2 & -\Xi_1 \\
\Sigma_2^\dagger & \Xi_1^\dagger & \Xi_2^\dagger
\end{pmatrix} \begin{pmatrix}
\Sigma_1 & \Sigma_2 \\
\Xi_2^\dagger & \Xi_1 \\
-\Xi_1^\dagger & \Xi_2
\end{pmatrix} = \begin{pmatrix}
\Sigma_1^\dagger \Sigma_1 + \Xi_2 \Xi_2^\dagger + \Xi_1 \Xi_1^\dagger & \Sigma_1^\dagger \Sigma_2 + \Xi_2 \Xi_1 - \Xi_1 \Xi_2 \\
\Sigma_2^\dagger \Sigma_1 + \Xi_1^\dagger \Xi_2^\dagger - \Xi_2^\dagger \Xi_1^\dagger & \Sigma_2^\dagger \Sigma_2 + \Xi_1^\dagger \Xi_1 + \Xi_2^\dagger \Xi_2
\end{pmatrix}.
\tag{2.209}
$$

For convenience, we introduce the $4k \times 4k$ matrix X and decompose the first ADHM equations as

$$\tilde{\Delta}^\dagger \tilde{\Delta} = \mathbf{1}_8 \otimes E_k^{(1,3)} \iff \left(\Delta^\dagger \Delta = \mathbf{1}_2 \otimes X_{[4k]} \text{ and } X = \mathbf{1}_4 \otimes E_k^{(1,3)} \right) \tag{2.210}$$

For above equation, (2.209) becomes

$$\begin{aligned}
\tilde{\Delta}^\dagger \tilde{\Delta} = \mathbf{1}_8 \otimes E_k^{(1,3)} &\iff \left(\Delta^\dagger \Delta = \mathbf{1}_2 \otimes X_{[4k]} \text{ and } X = \mathbf{1}_4 \otimes E_k^{(1,3)} \right) \\
&\iff \begin{cases} \Sigma_1^\dagger \Sigma_2 + \Xi_2 \Xi_1 - \Xi_1 \Xi_2 = 0, \\ \Sigma_1^\dagger \Sigma_1 + \Xi_2 \Xi_2^\dagger + \Xi_1 \Xi_1^\dagger = \Sigma_2^\dagger \Sigma_2 + \Xi_1^\dagger \Xi_1 + \Xi_2^\dagger \Xi_2, \\ \Sigma_1^\dagger \Sigma_1 + \Xi_2 \Xi_2^\dagger + \Xi_1 \Xi_1^\dagger = \mathbf{1}_4 \otimes E_k^{(1,3)}, \end{cases} \\
&\iff \begin{cases} [\Xi_1, \Xi_2] - \Sigma_1^\dagger \Sigma_2 = 0, \\ [\Xi_1, \Xi_1^\dagger] + [\Xi_2, \Xi_2^\dagger] + \Sigma_1^\dagger \Sigma_1 - \Sigma_2^\dagger \Sigma_2 = 0, \\ \Sigma_1^\dagger \Sigma_1 + \Xi_2 \Xi_2^\dagger + \Xi_1 \Xi_1^\dagger = \mathbf{1}_4 \otimes E_k^{(1,3)}. \end{cases}
\end{aligned} \tag{2.211}$$

First we expand the first equation of (2.211):

$$\begin{aligned}
&[\Xi_1, \Xi_2] - \Sigma_1^\dagger \Sigma_2 = 0 \iff \Xi_1 (\mathbf{1}_4 \otimes L_4) - (\mathbf{1}_4 \otimes L_4) \Xi_1 - \Sigma_1^\dagger \Sigma_2 = 0 \\
&\iff \begin{pmatrix} 0 & [L_3^\dagger, L_4] & -[L_2^\dagger, L_4] & -[L_1, L_4] \\ -[L_3^\dagger, L_4] & 0 & [L_1^\dagger, L_4] & -[L_2, L_4] \\ [L_2^\dagger, L_4] & -[L_1^\dagger, L_4] & 0 & -[L_3, L_4] \\ [L_1, L_4] & [L_2, L_4] & [L_3, L_4] & 0 \end{pmatrix} - \begin{pmatrix} S_1^\dagger S_5 & S_1^\dagger S_6 & S_1^\dagger S_7 & S_1^\dagger S_8 \\ S_2^\dagger S_5 & S_2^\dagger S_6 & S_2^\dagger S_7 & S_2^\dagger S_8 \\ S_3^\dagger S_5 & S_3^\dagger S_6 & S_3^\dagger S_7 & S_3^\dagger S_8 \\ S_4^\dagger S_5 & S_4^\dagger S_6 & S_4^\dagger S_7 & S_4^\dagger S_8 \end{pmatrix} = 0.
\end{aligned} \tag{2.212}$$

Next we expand the second equation of (2.211):

$$\begin{aligned}
&[\Xi_1, \Xi_1^\dagger] + [\Xi_2, \Xi_2^\dagger] + \Sigma_1^\dagger \Sigma_1 - \Sigma_2^\dagger \Sigma_2 = 0 \\
&\iff \begin{pmatrix} [L_3^\dagger, L_3] + [L_2^\dagger, L_2] + [L_1, L_1^\dagger] & [L_2, L_1^\dagger] - [L_2^\dagger, L_1] & [L_3, L_1^\dagger] - [L_3^\dagger, L_1] & [L_2, L_3] - [L_2^\dagger, L_3^\dagger] \\ [L_1, L_2^\dagger] - [L_1^\dagger, L_2] & [L_3^\dagger, L_3] + [L_1^\dagger, L_1] + [L_2, L_2^\dagger] & [L_3, L_2^\dagger] - [L_3^\dagger, L_2] & [L_3, L_1] - [L_3^\dagger, L_1^\dagger] \\ [L_1, L_3^\dagger] - [L_1^\dagger, L_3] & [L_2, L_3^\dagger] - [L_2^\dagger, L_3] & [L_2^\dagger, L_2] + [L_1^\dagger, L_1] + [L_3, L_3^\dagger] & -[L_2, L_1] + [L_2^\dagger, L_1^\dagger] \\ -[L_2^\dagger, L_3^\dagger] + [L_2, L_3] & [L_1^\dagger, L_3^\dagger] - [L_1, L_3] & -[L_1^\dagger, L_2^\dagger] + [L_1, L_2] & [L_1, L_1^\dagger] + [L_2, L_2^\dagger] + [L_3, L_3^\dagger] \end{pmatrix} \\
&\qquad + \mathbf{1}_4 \otimes [L_4, L_4^\dagger] + \begin{pmatrix} S_1^\dagger S_1 - S_5^\dagger S_5 & S_1^\dagger S_2 - S_5^\dagger S_6 & S_1^\dagger S_3 - S_5^\dagger S_7 & S_1^\dagger S_4 - S_5^\dagger S_8 \\ S_2^\dagger S_1 - S_6^\dagger S_5 & S_2^\dagger S_2 - S_6^\dagger S_6 & S_2^\dagger S_3 - S_6^\dagger S_7 & S_2^\dagger S_4 - S_6^\dagger S_8 \\ S_3^\dagger S_1 - S_7^\dagger S_5 & S_3^\dagger S_2 - S_7^\dagger S_6 & S_3^\dagger S_3 - S_7^\dagger S_7 & S_3^\dagger S_4 - S_7^\dagger S_8 \\ S_4^\dagger S_1 - S_8^\dagger S_5 & S_4^\dagger S_2 - S_8^\dagger S_6 & S_4^\dagger S_3 - S_8^\dagger S_7 & S_4^\dagger S_4 - S_8^\dagger S_8 \end{pmatrix} = 0.
\end{aligned} \tag{2.213}$$

Finally we expand the third equation of (2.211):

$$\begin{aligned}
&\Sigma_1^\dagger \Sigma_1 + \Xi_2 \Xi_2^\dagger + \Xi_1 \Xi_1^\dagger = \mathbf{1}_4 \otimes E_k^{(1,3)} \\
&\iff -\begin{pmatrix} -L_3^\dagger L_3 - L_2^\dagger L_2 - L_1 L_1^\dagger & [L_2^\dagger, L_1] & [L_3^\dagger, L_1] & [L_2^\dagger, L_3^\dagger] \\ [L_1^\dagger, L_2] & -L_3^\dagger L_3 - L_1^\dagger L_1 - L_2, L_2^\dagger & [L_3^\dagger, L_2] & [L_3^\dagger, L_1^\dagger] \\ [L_1^\dagger, L_3] & [L_2^\dagger, L_3] & -L_2^\dagger L_2 - L_1^\dagger L_1 - L_3 L_3^\dagger & -[L_2^\dagger, L_1^\dagger] \\ -[L_2, L_3] & [L_1, L_3] & -[L_1, L_2] & -L_1 L_1^\dagger - L_2 L_2^\dagger - L_3 L_3^\dagger \end{pmatrix} \\
&\qquad + \mathbf{1}_4 \otimes (L_4 L_4^\dagger) + \begin{pmatrix} S_1^\dagger S_1 & S_1^\dagger S_2 & S_1^\dagger S_3 & S_1^\dagger S_4 \\ S_2^\dagger S_1 & S_2^\dagger S_2 & S_2^\dagger S_3 & S_2^\dagger S_4 \\ S_3^\dagger S_1 & S_3^\dagger S_2 & S_3^\dagger S_3 & S_3^\dagger S_4 \\ S_4^\dagger S_1 & S_4^\dagger S_2 & S_4^\dagger S_3 & S_4^\dagger S_4 \end{pmatrix} = \mathbf{1}_4 \otimes E_k^{(1,3)}.
\end{aligned} \tag{2.214}$$

We obtain the following equation since the diagonal components of (2.214):

$$[L_2, L_2^\dagger] - [L_3, L_3^\dagger] + S_4^\dagger S_4 - S_1^\dagger S_1 = 0, \qquad \because [4,4] - [1,1] \text{ components} \tag{2.215a}$$

$$[L_3, L_3^\dagger] - [L_1, L_1^\dagger] + S_4^\dagger S_4 - S_2^\dagger S_2 = 0, \qquad \because [4,4] - [2,2] \text{ components} \tag{2.215b}$$

$$[L_1, L_1^\dagger] - [L_2, L_2^\dagger] + S_4^\dagger S_4 - S_3^\dagger S_3 = 0, \qquad \because [4,4] - [3,3] \text{ components} \tag{2.215c}$$

and since the off-diagonal components of (2.214):

$$[L_1, L_2^\dagger] + S_1^\dagger S_2 = 0, \qquad [L_1, L_2] + S_4^\dagger S_3 = 0, \qquad [L_1, L_3^\dagger] + S_1^\dagger S_3 = 0,$$
$$[L_2, L_3] + S_4^\dagger S_1 = 0, \qquad [L_2, L_3^\dagger] + S_2^\dagger S_3 = 0, \qquad [L_3, L_1] + S_4^\dagger S_2 = 0. \tag{2.216}$$

Insert this result into the $[1, 2]$ component of (2.213):

$$\begin{aligned}[L_2^\dagger, L_1] - [L_2, L_1^\dagger] - S_1^\dagger S_2 + S_5^\dagger S_6 = 0 &\iff [L_2, L_1^\dagger] - S_5^\dagger S_6 = 0, \qquad \because [L_1, L_2^\dagger] + S_1^\dagger S_2 = 0\\ &\iff \underbrace{\left([L_2, L_1^\dagger]\right)^\dagger}_{=[L_1,L_2^\dagger]=-S_1^\dagger S_2} - S_6^\dagger S_5 = 0,\\ &\iff S_6^\dagger S_5 = -S_1^\dagger S_2 \iff S_5^\dagger S_6 = -S_2^\dagger S_1.\end{aligned} \tag{2.217}$$

We take same calculation for the other off-diagonal components:

$$S_5^\dagger S_6 = -S_2^\dagger S_1, \qquad S_5^\dagger S_7 = -S_3^\dagger S_1, \qquad S_5^\dagger S_8 = -S_4^\dagger S_1,$$
$$S_6^\dagger S_7 = -S_3^\dagger S_2, \qquad S_6^\dagger S_8 = -S_4^\dagger S_2, \qquad S_7^\dagger S_8 = -S_4^\dagger S_3. \tag{2.218}$$

Next we consider the diagonal component of (2.216).

$$-[L_1, L_1^\dagger] + [L_2, L_2^\dagger] + [L_3, L_3^\dagger] - [L_4, L_4^\dagger] - S_1^\dagger S_1 + S_5^\dagger S_5 = 0, \tag{2.219a}$$
$$[L_1, L_1^\dagger] - [L_2, L_2^\dagger] + [L_3, L_3^\dagger] - [L_4, L_4^\dagger] - S_2^\dagger S_2 + S_6^\dagger S_6 = 0, \tag{2.219b}$$
$$[L_1, L_1^\dagger] + [L_2, L_2^\dagger] - [L_3, L_3^\dagger] - [L_4, L_4^\dagger] - S_3^\dagger S_3 + S_7^\dagger S_7 = 0, \tag{2.219c}$$
$$-[L_1, L_1^\dagger] - [L_2, L_2^\dagger] - [L_3, L_3^\dagger] - [L_4, L_4^\dagger] - S_4^\dagger S_4 + S_8^\dagger S_8 = 0, \tag{2.219d}$$

Now we can rewrite the two equations in the above equations to including only S term with using (2.215). For example, we take (2.219a) − (2.219b) and then using (2.215), thus we lead the first following equation:

$$2\left(S_2^\dagger S_2 - S_4^\dagger S_4\right) + S_3^\dagger S_3 - S_1^\dagger S_1 - S_7^\dagger S_7 + S_5^\dagger S_5 = 0, \qquad \because \text{(2.219a)} - \text{(2.219c)} \tag{2.220a}$$
$$2\left(S_4^\dagger S_4 - S_1^\dagger S_1\right) + S_3^\dagger S_3 - S_2^\dagger S_2 - S_7^\dagger S_7 + S_6^\dagger S_6 = 0, \qquad \because \text{(2.219b)} - \text{(2.219c)} \tag{2.220b}$$

Finally we consider (2.212), but cannot rewrite this equations in more simplify.

$$[L_4, L_1^\dagger] + S_2^\dagger S_7 = 0, \qquad [L_4, L_1] + S_4^\dagger S_5 = 0, \qquad [L_4, L_2^\dagger] + S_3^\dagger S_5 = 0,$$
$$[L_4, L_2] + S_4^\dagger S_6 = 0, \qquad [L_4, L_3^\dagger] + S_1^\dagger S_6 = 0, \qquad [L_4, L_3] + S_4^\dagger S_7 = 0. \tag{2.221}$$

and

$$S_1^\dagger S_5 = 0, \qquad S_2^\dagger S_6 = 0, \qquad S_3^\dagger S_7 = 0, \qquad S_4^\dagger S_8 = 0. \tag{2.222a}$$

$$S_2^\dagger S_7 = -S_3^\dagger S_6, \qquad S_4^\dagger S_5 = -S_1^\dagger S_8, \qquad S_3^\dagger S_5 = -S_1^\dagger S_7,$$
$$S_4^\dagger S_6 = -S_2^\dagger S_8, \qquad S_1^\dagger S_6 = -S_2^\dagger S_5, \qquad S_4^\dagger S_7 = -S_3^\dagger S_8. \tag{2.222b}$$

We now obtain the ADHM equations, and can roughly decompose these equations into two types: firstly the equations that combine $L(T)$ and S namely (2.215), (2.216), (2.219c), (2.219d) and (2.221), secondly the equations that contain only S namely (2.218), (2.220) and (2.222). We assume that the origin of the equations that contain only S parts is the outer parts of $S^\dagger S$ to the basis e_i. Let us prove this assumption as following. Thus we shall show that the assumption that S is able to expanded by the basis e_μ, namely $S^\dagger S = (e_\mu^\dagger \otimes \tilde{S}_\mu^\dagger)(e_\nu \otimes \tilde{S}_\nu)$, lead to (2.218), (2.220) and (2.222) as follows. Denote $M_1 := \tilde{S}_1 + i\tilde{S}_4,\ M_2 := \tilde{S}_2 + i\tilde{S}_5,\ M_3 := \tilde{S}_3 + i\tilde{S}_6,\ M_4 := \tilde{S}_8 + i\tilde{S}_7$ and $\overline{M}_1 := \tilde{S}_1 - i\tilde{S}_4,\ \overline{M}_2 := \tilde{S}_2 - i\tilde{S}_5,\ \overline{M}_3 := \tilde{S}_3 - i\tilde{S}_6,\ \overline{M}_4 := \tilde{S}_8 - i\tilde{S}_7$.

$$
\begin{aligned}
&(e_\mu^\dagger \otimes \tilde{S}_\mu^\dagger)(e_\nu \otimes \tilde{S}_\nu) \\
&= \begin{pmatrix}
\bar{M}_4^\dagger & 0 & 0 & 0 & 0 & -M_3^\dagger & M_2^\dagger & \bar{M}_1^\dagger \\
0 & \bar{M}_4^\dagger & 0 & 0 & M_3^\dagger & 0 & -M_1^\dagger & \bar{M}_2^\dagger \\
0 & 0 & \bar{M}_4^\dagger & 0 & -M_2^\dagger & M_1^\dagger & 0 & \bar{M}_3^\dagger \\
0 & 0 & 0 & \bar{M}_4^\dagger & -\bar{M}_1^\dagger & -\bar{M}_2^\dagger & -\bar{M}_3^\dagger & 0 \\
0 & -\bar{M}_3^\dagger & \bar{M}_2^\dagger & M_1^\dagger & M_4^\dagger & 0 & 0 & 0 \\
\bar{M}_3^\dagger & 0 & -\bar{M}_1^\dagger & M_2^\dagger & 0 & M_4^\dagger & 0 & 0 \\
-\bar{M}_2^\dagger & \bar{M}_1^\dagger & 0 & M_3^\dagger & 0 & 0 & M_4^\dagger & 0 \\
-M_1^\dagger & -M_2^\dagger & -M_3^\dagger & 0 & 0 & 0 & 0 & M_4^\dagger
\end{pmatrix}
\begin{pmatrix}
\bar{M}_4 & 0 & 0 & 0 & 0 & \bar{M}_3 & -\bar{M}_2 & -M_1 \\
0 & \bar{M}_4 & 0 & 0 & -\bar{M}_3 & 0 & \bar{M}_1 & -M_2 \\
0 & 0 & \bar{M}_4 & 0 & \bar{M}_2 & -\bar{M}_1 & 0 & -M_3 \\
0 & 0 & 0 & \bar{M}_4 & M_1 & M_2 & M_3 & 0 \\
0 & M_3 & -M_2 & -\bar{M}_1 & M_4 & 0 & 0 & 0 \\
-M_3 & 0 & M_1 & -\bar{M}_2 & 0 & M_4 & 0 & 0 \\
M_2 & -M_1 & 0 & -\bar{M}_3 & 0 & 0 & M_4 & 0 \\
\bar{M}_1 & \bar{M}_2 & \bar{M}_3 & 0 & 0 & 0 & 0 & M_4
\end{pmatrix} \\
&= \begin{pmatrix}
\mathcal{M}_1 & \bar{M}_1^\dagger\bar{M}_2 - M_2^\dagger M_1 & \bar{M}_1^\dagger\bar{M}_3 - M_3^\dagger M_1 & M_3^\dagger\bar{M}_2 - M_2^\dagger\bar{M}_3 & 0 & \bar{M}_4^\dagger\bar{M}_3 - M_3^\dagger M_4 & M_2^\dagger M_4 - \bar{M}_4^\dagger\bar{M}_2 & \bar{M}_1^\dagger M_4 - \bar{M}_4^\dagger M_1 \\
\bar{M}_2^\dagger\bar{M}_1 - M_1^\dagger M_2 & \mathcal{M}_2 & \bar{M}_2^\dagger\bar{M}_3 - M_3^\dagger M_2 & M_1^\dagger\bar{M}_3 - M_3^\dagger\bar{M}_1 & M_3^\dagger M_4 - \bar{M}_4^\dagger\bar{M}_3 & 0 & \bar{M}_4^\dagger\bar{M}_1 - M_1^\dagger M_4 & \bar{M}_2^\dagger M_4 - \bar{M}_4^\dagger M_2 \\
\bar{M}_3^\dagger\bar{M}_1 - M_1^\dagger M_3 & \bar{M}_3^\dagger\bar{M}_2 - M_2^\dagger M_3 & \mathcal{M}_3 & M_2^\dagger\bar{M}_1 - M_1^\dagger\bar{M}_2 & \bar{M}_4^\dagger\bar{M}_2 - M_2^\dagger M_4 & M_1^\dagger M_4 - \bar{M}_4^\dagger\bar{M}_1 & 0 & \bar{M}_3^\dagger M_4 - \bar{M}_4^\dagger M_1 \\
\bar{M}_2^\dagger M_3 - \bar{M}_3^\dagger M_2 & \bar{M}_3^\dagger M_1 - \bar{M}_1^\dagger M_3 & \bar{M}_1^\dagger M_2 - \bar{M}_2^\dagger M_1 & \mathcal{M}_4 & \bar{M}_4^\dagger M_1 - \bar{M}_1^\dagger M_4 & \bar{M}_4^\dagger M_2 - \bar{M}_2^\dagger M_4 & \bar{M}_4^\dagger M_3 - \bar{M}_3^\dagger M_4 & 0 \\
0 & M_4^\dagger M_3 - \bar{M}_4\bar{M}_3^\dagger & \bar{M}_2^\dagger\bar{M}_4 - M_4^\dagger M_2 & M_1^\dagger\bar{M}_4 - M_4^\dagger\bar{M}_1 & \mathcal{M}_5 & M_1^\dagger M_2 - \bar{M}_2^\dagger\bar{M}_1 & M_1^\dagger M_3 - \bar{M}_3^\dagger\bar{M}_1 & \bar{M}_3^\dagger M_2 - \bar{M}_2^\dagger M_3 \\
\bar{M}_3^\dagger\bar{M}_4 - M_4^\dagger M_3 & 0 & M_4^\dagger M_1 - \bar{M}_1^\dagger\bar{M}_4 & M_2^\dagger\bar{M}_4 - M_4^\dagger\bar{M}_2 & M_2^\dagger M_1 - \bar{M}_1^\dagger\bar{M}_2 & \mathcal{M}_6 & M_2^\dagger M_3 - \bar{M}_3^\dagger\bar{M}_2 & \bar{M}_1^\dagger M_3 - \bar{M}_3^\dagger M_1 \\
M_4^\dagger M_2 - \bar{M}_2^\dagger\bar{M}_4 & \bar{M}_1^\dagger\bar{M}_4 - M_4^\dagger M_1 & 0 & M_3^\dagger\bar{M}_4 - M_4^\dagger\bar{M}_3 & M_3^\dagger M_1 - \bar{M}_1^\dagger\bar{M}_3 & M_3^\dagger M_2 - \bar{M}_2^\dagger\bar{M}_3 & \mathcal{M}_7 & \bar{M}_2^\dagger M_1 - \bar{M}_1^\dagger M_2 \\
M_4^\dagger\bar{M}_1 - M_1^\dagger\bar{M}_4 & M_4^\dagger\bar{M}_2 - M_2^\dagger\bar{M}_4 & M_4^\dagger\bar{M}_3 - M_3^\dagger\bar{M}_4 & 0 & M_2^\dagger\bar{M}_3 - M_3^\dagger\bar{M}_2 & M_3^\dagger\bar{M}_1 - M_1^\dagger\bar{M}_3 & M_1^\dagger\bar{M}_2 - M_2^\dagger\bar{M}_1 & \mathcal{M}_8
\end{pmatrix}
\end{aligned}
\tag{2.223a}
$$

$$
\text{where}
\begin{cases}
\mathcal{M}_1 \coloneqq M_1^\dagger M_1 + M_2^\dagger M_2 + M_3^\dagger M_3 + \bar{M}_4^\dagger\bar{M}_4, & \mathcal{M}_2 \coloneqq M_1^\dagger M_1 + \bar{M}_2^\dagger\bar{M}_2 + M_3^\dagger M_3 + \bar{M}_4^\dagger\bar{M}_4, \\
\mathcal{M}_3 \coloneqq M_1^\dagger M_1 + M_2^\dagger M_2 + \bar{M}_3^\dagger\bar{M}_3 + \bar{M}_4^\dagger\bar{M}_4, & \mathcal{M}_4 \coloneqq \bar{M}_1^\dagger\bar{M}_1 + \bar{M}_2^\dagger\bar{M}_2 + \bar{M}_3^\dagger\bar{M}_3 + \bar{M}_4^\dagger\bar{M}_4, \\
\mathcal{M}_5 \coloneqq M_1^\dagger M_1 + \bar{M}_2^\dagger\bar{M}_2 + \bar{M}_3^\dagger M_3 + M_4^\dagger M_4, & \mathcal{M}_6 \coloneqq \bar{M}_1^\dagger\bar{M}_1 + M_2^\dagger M_2 + \bar{M}_3^\dagger\bar{M}_3 + M_4^\dagger M_4, \\
\mathcal{M}_7 \coloneqq \bar{M}_1^\dagger\bar{M}_1 + \bar{M}_2^\dagger\bar{M}_2 + M_3^\dagger M_3 + M_4^\dagger M_4, & \mathcal{M}_8 \coloneqq M_1^\dagger M_1 + M_2^\dagger M_2 + M_3^\dagger M_3 + M_4^\dagger M_4.
\end{cases}
\tag{2.223b}
$$

$$
\equiv \begin{pmatrix}
S_1^\dagger S_1 & S_1^\dagger S_2 & S_1^\dagger S_3 & S_1^\dagger S_4 & S_1^\dagger S_5 & S_1^\dagger S_6 & S_1^\dagger S_7 & S_1^\dagger S_8 \\
S_2^\dagger S_1 & S_2^\dagger S_2 & S_2^\dagger S_3 & S_2^\dagger S_4 & S_2^\dagger S_5 & S_2^\dagger S_6 & S_2^\dagger S_7 & S_2^\dagger S_8 \\
S_3^\dagger S_1 & S_3^\dagger S_2 & S_3^\dagger S_3 & S_3^\dagger S_4 & S_3^\dagger S_5 & S_3^\dagger S_6 & S_3^\dagger S_7 & S_3^\dagger S_8 \\
S_4^\dagger S_1 & S_4^\dagger S_2 & S_4^\dagger S_3 & S_4^\dagger S_4 & S_4^\dagger S_5 & S_4^\dagger S_6 & S_4^\dagger S_7 & S_4^\dagger S_8 \\
S_5^\dagger S_1 & S_5^\dagger S_2 & S_5^\dagger S_3 & S_5^\dagger S_4 & S_5^\dagger S_5 & S_5^\dagger S_6 & S_5^\dagger S_7 & S_5^\dagger S_8 \\
S_6^\dagger S_1 & S_6^\dagger S_2 & S_6^\dagger S_3 & S_6^\dagger S_4 & S_6^\dagger S_5 & S_6^\dagger S_6 & S_6^\dagger S_7 & S_6^\dagger S_8 \\
S_7^\dagger S_1 & S_7^\dagger S_2 & S_7^\dagger S_3 & S_7^\dagger S_4 & S_7^\dagger S_5 & S_7^\dagger S_6 & S_7^\dagger S_7 & S_7^\dagger S_8 \\
S_8^\dagger S_1 & S_8^\dagger S_2 & S_8^\dagger S_3 & S_8^\dagger S_4 & S_8^\dagger S_5 & S_8^\dagger S_6 & S_8^\dagger S_7 & S_8^\dagger S_8
\end{pmatrix} = S^\dagger S.
\tag{2.223c}
$$

Now we compare (2.223a) with (2.223c), and then we obtain the following relations.

$$
S_1^\dagger S_5 = S_2^\dagger S_6 = S_3^\dagger S_7 = S_4^\dagger S_8 = 0, \tag{2.224}
$$

these relations are just (2.222a) and

$$
\begin{aligned}
&S_1^\dagger S_2 = -S_6^\dagger S_5, && S_1^\dagger S_4 = -S_8^\dagger S_5, && S_2^\dagger S_5 = -S_1^\dagger S_6, && S_3^\dagger S_6 = -S_2^\dagger S_7, \\
&S_1^\dagger S_3 = -S_7^\dagger S_5, && S_2^\dagger S_4 = -S_8^\dagger S_6, && S_3^\dagger S_5 = -S_1^\dagger S_7, && S_4^\dagger S_6 = -S_2^\dagger S_8, \\
&S_2^\dagger S_3 = -S_7^\dagger S_6, && S_3^\dagger S_4 = -S_8^\dagger S_7, && S_4^\dagger S_5 = -S_1^\dagger S_8, && S_4^\dagger S_7 = -S_3^\dagger S_8,
\end{aligned}
\tag{2.225}
$$

these relations are just (2.218) and (2.222b). For (2.223b) (and (2.223c)), we obtatin (2.220) also.

Therefore we found that the first ADHM equations more simplify when suppose $S = e_\mu \otimes \tilde{S}_\mu$. Now the obtained first ADHM equations is complex representations, thus we rewritten to the real representations by rewriting L to T. Moreover we note that $[T^\mu, T^\nu]$ becomes $[T^\mu, T^\nu]$ because we can drop the x^μ terms which are the identity matrix $\mathbf{1}_8$. Hence we obtain the first ADHM equations under supposing $S = e_\mu \otimes \tilde{S}_\mu$:

$$[T^2, T^5] - [T^3, T^6] + \frac{i}{2}\left(S_4^\dagger S_4 - S_1^\dagger S_1\right) = 0,$$
$$[T^3, T^6] - [T^1, T^4] + \frac{i}{2}\left(S_4^\dagger S_4 - S_2^\dagger S_2\right) = 0,$$
$$[T^1, T^4] - [T^2, T^5] + \frac{i}{2}\left(S_4^\dagger S_4 - S_3^\dagger S_3\right) = 0, \tag{2.226}$$

$$[T^1, T^4] + [T^2, T^5] - [T^3, T^6] - [T^8, T^7] - \frac{i}{2}\left(S_3^\dagger S_3 - S_7^\dagger S_7\right) = 0,$$
$$-[T^1, T^4] - [T^2, T^5] - [T^3, T^6] - [T^8, T^7] - \frac{i}{2}\left(S_4^\dagger S_4 - S_8^\dagger S_8\right) = 0, \tag{2.227}$$

$$\begin{aligned}
&[T^1, T^2] + [T^4, T^5] + \frac{1}{2}\left(S_1^\dagger S_2 - S_2^\dagger S_1\right) = 0, && [T^1, T^5] - [T^4, T^2] + \frac{i}{2}\left(S_1^\dagger S_2 + S_2^\dagger S_1\right) = 0,\\
&[T^1, T^3] + [T^4, T^6] + \frac{1}{2}\left(S_1^\dagger S_3 - S_3^\dagger S_1\right) = 0, && [T^1, T^6] - [T^4, T^3] + \frac{i}{2}\left(S_1^\dagger S_3 + S_3^\dagger S_1\right) = 0,\\
&[T^2, T^3] + [T^5, T^6] + \frac{1}{2}\left(S_2^\dagger S_3 - S_3^\dagger S_2\right) = 0, && [T^2, T^6] - [T^5, T^3] + \frac{i}{2}\left(S_2^\dagger S_3 + S_3^\dagger S_2\right) = 0,\\
&[T^1, T^2] - [T^4, T^5] + \frac{1}{2}\left(S_4^\dagger S_3 - S_3^\dagger S_4\right) = 0, && [T^1, T^5] + [T^4, T^2] - \frac{i}{2}\left(S_4^\dagger S_3 + S_3^\dagger S_4\right) = 0,\\
&[T^2, T^3] - [T^5, T^6] + \frac{1}{2}\left(S_4^\dagger S_1 - S_1^\dagger S_4\right) = 0, && [T^2, T^6] + [T^5, T^3] - \frac{i}{2}\left(S_4^\dagger S_1 + S_1^\dagger S_4\right) = 0,\\
&[T^3, T^1] - [T^6, T^4] + \frac{1}{2}\left(S_4^\dagger S_2 - S_2^\dagger S_4\right) = 0, && [T^3, T^4] + [T^6, T^1] - \frac{i}{2}\left(S_4^\dagger S_2 + S_2^\dagger S_4\right) = 0,\\
&[T^8, T^1] + [T^7, T^4] + \frac{1}{2}\left(S_2^\dagger S_7 - S_7^\dagger S_2\right) = 0, && [T^8, T^4] - [T^7, T^1] + \frac{i}{2}\left(S_2^\dagger S_7 + S_7^\dagger S_2\right) = 0,\\
&[T^8, T^2] + [T^7, T^5] + \frac{1}{2}\left(S_3^\dagger S_5 - S_5^\dagger S_3\right) = 0, && [T^8, T^5] - [T^7, T^2] + \frac{i}{2}\left(S_3^\dagger S_5 + S_5^\dagger S_3\right) = 0,\\
&[T^8, T^3] + [T^7, T^6] + \frac{1}{2}\left(S_1^\dagger S_6 - S_6^\dagger S_1\right) = 0, && [T^8, T^6] - [T^7, T^3] + \frac{i}{2}\left(S_1^\dagger S_6 + S_6^\dagger S_1\right) = 0,\\
&[T^8, T^1] - [T^7, T^4] + \frac{1}{2}\left(S_4^\dagger S_5 - S_5^\dagger S_4\right) = 0, && [T^8, T^4] + [T^7, T^1] - \frac{i}{2}\left(S_4^\dagger S_5 + S_5^\dagger S_4\right) = 0,\\
&[T^8, T^2] - [T^7, T^5] + \frac{1}{2}\left(S_4^\dagger S_6 - S_6^\dagger S_4\right) = 0, && [T^8, T^5] + [T^7, T^2] - \frac{i}{2}\left(S_4^\dagger S_6 + S_6^\dagger S_4\right) = 0,\\
&[T^8, T^3] - [T^7, T^6] + \frac{1}{2}\left(S_4^\dagger S_7 - S_7^\dagger S_4\right) = 0, && [T^8, T^6] + [T^7, T^3] - \frac{i}{2}\left(S_4^\dagger S_7 + S_7^\dagger S_4\right) = 0.
\end{aligned} \tag{2.228}$$

Next we lead the ADHM equations which are associated with the second ADHM constraint, we call these equations as "the second ADHM equations". It is straightforward that we lead the second ADHM equations. The equation (2.43) stand without change:

$$fT^\mu = T^\mu f. \tag{2.229}$$

The equation (2.44) is satisfied if and only if the following equations is satisfied:

$$[T^\mu, T^\nu] = 0. \tag{2.230}$$

Note that I mistaken the second ADHM equations in [25], there is an omission of some equations.

2.7 Higher dimensional calorons

In this subsection, we consider higher-dimensional calorons and the monopole limit. It is well known that the Harrington-Shepard (HS) one-caloron in the four dimensions can be generated by the 't Hooft multi-instantons that are periodic in one of the four coordinates [13]. Can we generate a higher-dimensional HS type one-caloron with the same method in the four dimensions? Let us discuss this question in the following. We will use the multi-instantons to produce the HS type caloron. However, the 't Hooft type multi-instantons in the higher dimensions are well-defined only if we assume the well-separated limit (2.68). Therefore we use the 't Hooft type multi-instanton with well-separated on the periodic coordinate direction $t = x^{4n}$ to produce the higher-dimensional caloron.

We consider the situation that same size 't Hooft type one-instantons are lined up on the x^{4n}-direction with well-separated. This gauge field is given by

$$A_\mu(x) = \frac{1}{4}\Sigma^{(\pm)}_{\mu\nu}\partial_\nu \ln \phi_{\text{'t Hooft}}(x), \tag{2.231}$$

where

$$\phi_{\text{'t Hooft}}(x) = 1 + \sum_{p=-P}^{P} \frac{\lambda^2}{||\mathbf{x} - \mathbf{a}_x||^2 + (x^{4n} - a_p^{4n})^2}. \tag{2.232}$$

Here $\lambda \in \mathbb{R}$ is the instanton size, $\mathbf{x} = x^i$ ($i = 1, 2, \ldots, 4n-1$), $\mathbf{a}_x \in \mathbb{R}^{4n-1}$ is the instanton's position (without the x^{4n}-direction) and $a_p^{4n} \in \mathbb{R}$ is the positions on the x^{4n}-direction. For the well-separated limit (2.68), the x^{4n}-direction position a_p^{4n} satisfies the condition: $(a_p^{4n} - a_q^{4n})^2 \gg \lambda^2$ ($p \neq q$).

Now we choose the x^{4n}-direction positions $a_p^{4n} = a_t - p\beta$ ($a_t, \beta \in \mathbb{R}$), and we take the limit $P \to \infty$ and the x^{4n}-direction to periodic direction with the periodicity β as $\mathbb{R}^{4n} \to \mathbb{R}^{4n-1} \times S^1$. This situation replace the well-separated limit ($(a_p^{4n} - a_{p+1}^{4n})^2 \gg \lambda^2$ for all p) to the condition of the size λ and the periodicity β:

$$\beta \gg \lambda. \tag{2.233}$$

In addition, $\phi_{\text{'t Hooft}}(x)$ becomes

$$\begin{aligned}\lim_{p\to\infty} \phi_{\text{'t Hooft}}(x) &= 1 + \sum_{p=-\infty}^{\infty} \frac{\lambda^2}{||\mathbf{x} - \mathbf{a}_x||^2 + (t - (a_t - p\beta))^2} \\ &= 1 + \mu^2\lambda^2 \sum_{p=-\infty}^{\infty} \frac{1}{\mu^2||\mathbf{x} - \mathbf{a}_x||^2 + (\mu(t - a_t) + 2\pi p)^2},\end{aligned} \tag{2.234}$$

where $\mu = 2\pi/\beta$. Note that we demand the condition $2\pi \gg \mu^2\lambda^2$ from $\beta^2 \gg \lambda^2$, but this condition does not have an influence on that we take the factor μ^{-2} from the dominator. Now we use the formula:

$$\sum_{p=-\infty}^{\infty} \frac{1}{a^2 + (b + 2\pi p)^2} = \frac{\sinh a}{2a(\cosh a - \cos b)}, \tag{2.235}$$

then a gauge field of the HS type one-caloron in the higher dimensions ($n \geq 2$) is given by

$$A_\mu = \frac{1}{4}\Sigma^{(\pm)}_{\mu\nu}\partial_\nu \ln\left(1 + \frac{\pi\lambda^2 \sinh(2\pi r/\beta)}{\beta r\,(\cosh(2\pi r/\beta) - \cos(2\pi\tilde{t}/\beta))}\right), \quad \text{with } \beta \gg \lambda. \tag{2.236}$$

Here $r = \sqrt{(x^i - a^i)^2}$, $\tilde{t} = x^{4n} - a^{4n}$, for any $a^{4n} \in [0, \beta)$ and the index $i = 1, \ldots, 4n-1$. The condition $\beta \gg \lambda$ means that the caloron's size modulus λ is much smaller than the periodic coordinate size β, hence we call this condition (2.233) as the small size limit.

In four dimensions, the HS one-caloron becomes the gauge-equivalent to the BPS one-monopole when we take the limit $\beta/2\pi\lambda \to 0$ [41, 42]. On the other hand, in the higher dimensions, the HS type one-caloron requires the small size limit $\beta/\lambda \gg 1$, hence the monopole limit $\beta/2\pi\lambda \to 0$ is evidently inconsistent with this limit. Therefore we can not take the monopole limit for the HS type one-caloron in higher dimensions.

Part II

The Atiyah-Manton construction and Skyrmions

Chapter 3

The Skyrme model in four dimensions

In this chapter, we review the Skyrme model in four dimensions.

The Skyrme model [8] is a model for hadrons in the low-energy effective theory of QCD. The model is a four-dimensional non-linear sigma model whose target space is $S^3 \sim \mathrm{SU}(2)$, and composed of the fourth order derivative term in addition to the canonical kinetic term. The fourth order derivative term guarantees the stability of solitons of co-dimension three, which are called Skyrmions. The Skyrmions are characterized by the homotopy class $\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$ and they are regarded as Baryons. The energy functional of the Skyrme model has the Bogomol'nyi bound given by the topological charge associated with the homotopy. This topological charge is identified with the Baryon number. However, no analytic solutions that saturate the lower bound of the energy have been found so far [1]. There have only been obtained the numerical solutions of Skyrmions, which indeed exceed the energy bound. This reflects the fact that the original four dimensional Skyrme model does not have the BPS property.

Finding proper solutions of Skyrmions is a long standing problem. There are several directions to construct solutions. For example, the rational map ansatz provides a good approximation to the Skyrmion solutions [46]. This includes solutions corresponding to higher Baryon numbers. Although they can not saturate the energy bound, the rational map solutions have close energies to the normalized Baryon charges. Alternatively, there is another promising approach to Skyrmions known as the Atiyah-Manton construction [10]. Atiyah and Manton pointed out that the holonomy of the Yang-Mills instantons in the four-dimensional Euclid space [2] gives a well approximated static Skyrmion solutions. Although, the origin of this approximation is not transparent, a physical interpretation to the Atiyah-Manton construction of Skyrmions was discussed in [48, 49].

Even though the Skyrmion solutions are well-approximated by instantons, they never saturate the Bogomol'nyi bound of the energy. In order to understand the obscure connection between the Yang-Mills instantons and Skyrmions, we need further penetrating analysis. In this context, inspired by a holographic QCD model [51], it is proposed a systematic derivation of the energy functional for the static Skyrme field from the Yang-Mills action in four dimensions [50]. In the derivation, the introduction of the tower of mesons originated from the Kaluza-Klein-like expansion modes in higher dimensions makes the Atiyah-Manton solution have closer energy to the bound [52]. Therefore, including the higher expansion modes in the Atiyah-Manton solution leads to the better approximation to the Skyrmions. Moreover, this relation is generalized to lower dimensions. For example, an analogue of the Atiyah-Manton construction in two dimensions is proposed [53, 54] where the sine-Gordon soliton solution in one dimensions is well-approximated by the $\mathbb{C}P^1$-lump – the two-dimensional instantons. These facts remarkably suggest that there is a deep correspondence between instantons or solitons and Skyrmion-like objects in various dimensions.

The organization of this chapter as follows. Section 3.1 is introduced the Skyrme model and single Skyrmion. Section 3.2 is reviewed the Sutcliffe's truncation method which is leaded the Skyrme model from the (pure-)Yang-Mills action. Sectoin 3.3 is reviewed the Atiyah-Manton construcion.

3.1 Skyrme model and hedgehog ansatz

The action of Skyrme model is defined by

$$S_{\mathrm{Sky}} = \int d^3x dt \left(-\frac{f_\pi^2}{4} \mathrm{Tr}\left(R^\mu R_\mu\right) + \frac{1}{32e^2} \mathrm{Tr}\left([R_\mu, R_\nu]^2\right) \right). \tag{3.1}$$

[1] This is not the case for Skyrme models in curved spaces. For example, see [43, 44, 45] and references therein.

[2] The case for the curved spaces was discussed in [47].

where $\mu, \nu = 1, 2, 3, 4$, $R_\mu = U\partial_\mu U^{-1}$ and $U(x) \in \mathrm{SU}(2)$ is called as the Skyrme field. The dimensionless constant e is called as Skyrme parameter, and f_π is pion decay constant which dimension is $[f_\pi] = [L^{-1}]$. Now we take the rescalings of the length $x^\mu \to (f_\pi e)^{-1} x^\mu$ and the overall factor of the Lagrangian $L \to \frac{f_\pi}{2e} L$, then we obtain the dimensionless Skyrme model:

$$S_{\text{Sky}} = \int d^3x dt \left(-\frac{1}{2} \mathrm{Tr}\left(R^\mu R_\mu\right) + \frac{1}{16} \mathrm{Tr}\left([R_\mu, R_\nu]^2\right) \right). \tag{3.2}$$

The Euler-Lagrange equation of this Lagrangian is

$$\partial_\mu \left(R^\mu + \frac{1}{4} [R^\nu, [R_\nu, R^\mu]] \right) = 0. \tag{3.3}$$

Although this time dependent Lagrangian is original model which was given by Skyrme, we consider the time independent Skyrme model which has topological soliton, known as Skyrmion.

The time independen Skyrme model is given by

$$E_{\text{Skyrme}} = \int d^3x \left(-\frac{1}{2} \mathrm{Tr}[R_i R_i] - \frac{1}{16} \mathrm{Tr}[R_i, R_j]^2 \right). \tag{3.4}$$

We call this static Lagrangian (energy functional) as the (static) Skyrme model in the following. The Bogomol'nyi completion of the energy functional (3.4) gives the energy bound $E_{\text{Skyrme}} \geq 12\pi^2 |B|$ where B is the topological charge, namely, the Baryon number:

$$B = -\frac{1}{24\pi^2} \int d^3x \, \varepsilon_{ijk} \mathrm{Tr}[R_i R_j R_k], \tag{3.5}$$

where ε_{ijk} is the Levi-Civita symbol. This Bogomolol'nyi completion is easily shown as follows:

$$\begin{aligned} \mathrm{Tr}\left(\frac{1}{\sqrt{2}} R_i \pm \frac{1}{2\sqrt{2}} \varepsilon_{ijk} R_j R_k \right)^2 \geq 0 \iff & \mathrm{Tr}\left(\frac{1}{2} R_i R_i + \frac{1}{8} (\delta^j_l \delta^k_m - \delta^j_m \delta^k_l) R_j R_k R^l R^m \right) \pm 2 \frac{1}{4} \varepsilon_{ijk} \mathrm{Tr}\left(R_i R_j R_k\right) \geq 0 \\ \iff & \frac{1}{2} \mathrm{Tr}\left(R_i R_i\right) + \frac{1}{16} \mathrm{Tr}\left([R_i, R_j]^2\right) \geq \mp \frac{1}{2} \varepsilon_{ijk} \mathrm{Tr}\left(R_i R_j R_k\right), \end{aligned} \tag{3.6}$$

where we using $\varepsilon^{ijk}\varepsilon_{ilm} = \delta^j_l \delta^k_m - \delta^j_m \delta^k_l$ and $R_j R_k R^j R^k - R_j R_k R^k R^j = \frac{1}{2}(R_j R_k - R_k R_j)^2 = \frac{1}{2}[R_j, R_k]^2$. The equation of motion derived from (3.4) is

$$\partial_i \left(R_i - \frac{1}{4} [R_j, [R_j, R_i]] \right) = 0. \tag{3.7}$$

No analytic solutions to this equation (3.7) have been found but a spherically symmetric solution is dealt with the following hedgehog ansatz:

$$U = \exp\left(i f(r) \hat{x}^i \sigma_i \right). \tag{3.8}$$

Here $\hat{x}^i = \frac{x^i}{r}$, $r^2 = x^i x^i$ and σ_i are the Pauli matrices, namely, the quaternion basis. The energy functional for this ansatz is evaluated to be

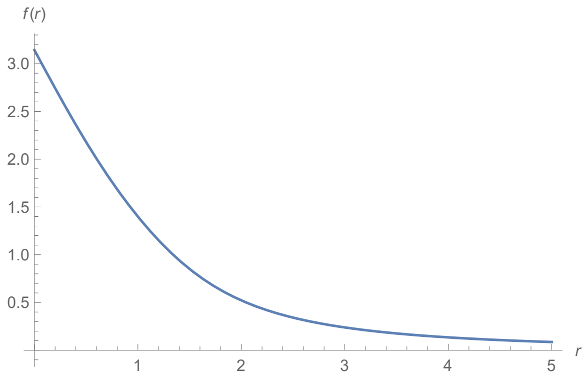
$$E_{\text{Skyrme}} = \int_0^\infty dr \int_{S^2} d\Omega_2 \mathcal{E}(r) = 2\pi \int_0^\infty dr \left(r^2 (\partial_r f)^2 + 2\sin^2 f \left(1 + (\partial_r f)^2\right) + \frac{\sin^4 f}{r^2} \right). \tag{3.9}$$

Here $\mathcal{E}(r)$ is the energy density and $d\Omega_2$ is the integral element of the two-dimensional sphere. The equation of motion with hedgehog ansatz, sometimes we call this equation as the Hedgehog equation, is

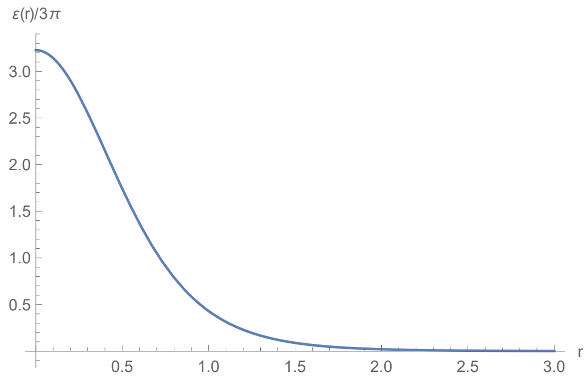
$$(r^2 + 2\sin^2 f) \partial_r^2 f + 2r \partial_r f + \sin 2f \left((\partial_r f)^2 - 1 - \frac{\sin^2 f}{r^2} \right) = 0. \tag{3.10}$$

Moreover, substitute the hedgehog ansatz (3.8) into the topological charge (3.5), then we obtain

$$B = \frac{1}{\pi} \left(f(0) - f(\infty) \right). \tag{3.11}$$



(a) The profile for 4d Skyrmion.



(b) The energy density plot.

Figure 3.1: The numerical profile for $f(r)$ and the plot for the energy density $\mathcal{E}(r)$.

Now we consider a single Skyrmion, namely the $B = 1$ Skyrmion, thus the boundary condition is given by $f(0) = \pi$, $f(\infty) = 0$. The numerical study is easily performed for this ansatz. The solution to the equation of motion (3.7) with the ansatz (3.8) is found in Fig. 3.1. We insert this numerical result into the energy functional (3.9), and then the energy is calculated to be $E = 1.2314 \times 12\pi^2$, to four decimal place. Therefore the $B = 1$ Skyrmion exceeds the Bogomol'nyi bound, sometimes called BPS bound, by approvimately 23%, namely the Skyrmion is not BPS soliton.

In the following, we will mention how the existence of the numerical solutions in (3.10) is guaranteed. For later convenience, we now rewrite the Hedgehog equation (3.10) by multiplying r^2 on both side:

$$r^2(r^2 + 2\sin^2 f)\partial_r^2 f + 2r^3\partial_r f + \sin 2f\left(r^2(\partial_r f)^2 - r^2 - \sin^2 f\right) = 0. \tag{3.12}$$

Consider analysis of the profile function at the origin [3], to examine the existence of the numerical solutions. Let the profile function be smooth, namely $f(r)$ is class C^∞, and then the Tayler expansion around origin becomes $f(\delta r) = \sum_{m=0}^{\infty} f_m(\delta r)^m = f_0 + f_1\delta r + f_2(\delta r)^2 + \ldots$. Using the chain rule, the expansion of $\sin f(r)$ around origin becomes

$$\sin f(\delta r) = \sum_{m=0}^{\infty} \frac{\partial_r^m \sin f(r)}{m!}\Big|_{r=0}(\delta r)^m = \sin f_0 + \cos f_0 \cdot f_1\delta r + \frac{1}{2}\left(-\sin f_0 \cdot f_1^2 + \cos f_0 \cdot 2f_2\right)(\delta r)^2 + \ldots. \tag{3.13}$$

After similarly calculate the expansion of $\sin 2f(r)$ around origin, we write down the equations for the coefficients f_m by substituting these expansion and decide f_m for the equations of each order $(\delta r)^m$ to be zero. Specifically, the zero order $(\delta r)^0$ becomes

$$\sin 2f_0 \sin^2 f_0 = 0 \iff \begin{cases} f_0 = \left(\frac{1}{2} + n\right)\pi, \\ f_0 = n\pi. \end{cases} \tag{3.14}$$

The first order δr becomes

$$(-\cos 2f_0 + \cos 4f_0)\, f_1 = 0, \tag{3.15}$$

Now we take $f_0 = f(0) = \pi$ to consider the single Skyrmion, thus f_1 is free parameter. In the following, we take these conditions. The second and third orders $(\delta r)^2, (\delta r)^3$ become zero automatically. The fourth order $(\delta r)^4$ becomes

$$\left(1 + 2f_1^2\right) f_2 = 0, \tag{3.16}$$

thus we obtain $f_2 = 0$. The fifth order $(\delta r)^5$ becomes

$$2f_1^3 + f_1^5 + 15(1 + 2f_1^2)f_3 = 0 \iff f_3 = -\frac{2f_1^3 + f_1^5}{15\left(1 + 2f_1^2\right)}. \tag{3.17}$$

[3]Strictly speaking, we have to consider the analysis at the infinity also, but now we treat only the origin for simplify.

Similarly we can decide the higher order coefficients f_m by calculating order by order in $(\delta r)^p$, and then the expansion of the profile function around the origin is given by

$$f(\delta r) = \pi + f_1 \delta r - \frac{2f_1^3 + f_1^5}{15\left(1+2f_1^2\right)}(\delta r)^3 + \frac{f_1^5\left(10+16f_1^2+11f_1^4+14f_1^6\right)}{350\left(1+2f_1^2\right)^2}(\delta r)^5 + O\left((\delta r)^7\right). \tag{3.18}$$

For this result, we find that the higher order coefficients f_m $(m \geq 2)$ is written by lower order coefficients f_0 and f_1. Note that we have been taking $f_0 = \pi$ thus f_0 does not appear explicitly in the expansion (3.18), but the higher order coefficients essentially depend f_0 also. By the way, the nth ordinary differential equation need n boundary conditions to decide solution uniquely. In other words, it is needed that n free parameters existence when the series expansions of unknown function. Now recall that the hedgehog equation is second order ordinary differential equation, thus the series expansions need two free parameters. For (3.18), f_0 and f_1 are free parameters thus this condition is satisfied. This fact guarantees the existence of (numerical) solution in (3.10). In contrast to this, if above free parameters condition is not satisfied then a solution with the boundary condition does not existence in generally.

3.2 Overview of the Sutcliffe's truncation in four dimensions

The four-dimensional energy functional for static fields[4] of the Skyrme model is obtained by a reduction of the usual quadratic Yang-Mills action in the four-dimensional Euclidean space. The action is

$$S = -\frac{1}{2\kappa g^2}\int \mathrm{Tr}[*_4 F \wedge F] = -\frac{1}{4\kappa g^2}\int d^4x\, \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]. \tag{3.19}$$

Here $F = \frac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu$, $(\mu,\nu = 1,\dots,4)$ is the gauge field strength 2-form. The component is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The gauge field A_μ is in the adjoint representation of a gauge group G and it is expanded by the generators T^a $(a = 1,\dots \dim\mathcal{G})$. Here $\mathcal{G}$ is the Lie algebra associated with G and κ is the normalization constant for the generators $\mathrm{Tr}[T^aT^b] = \kappa\delta^{ab}$. Here g is the gauge coupling constant. Making the action (3.19) be the completely square form results in the Bogomol'nyi-Prasad-Sommerfield (BPS) self-duality equation $F = *_4 F$ whose solutions are called instantons (see section 1). Since the Yang-Mills action (3.19) has the scale invariance, instanton solutions that saturate the Bogomol'nyi bound have a size modulus.

It is proposed in [50] that a holography-inspired reduction of the four-dimensional Yang-Mills action (3.19) provides the energy functional for the static Skyrme field. Following the prescription in [50], we first decompose the four-dimensional Euclidean space into the three-dimensional physical space and a "fictious" direction: $x^\mu = (x^i, x^4)$ where $i = 1,\dots,3$. We then expand the four-dimensional gauge field $A_\mu(x^i, x^4)$ in the *infinite line* along the x^4-direction by a complete orthonormal basis with the square integrable function. A suitable basis with the boundary condition $A_i(x^i, x^4) \to 0$ as $x^4 \to \infty$ is a Hermite function [5]:

$$\psi_m(z) = \frac{(-1)^m}{\sqrt{m!2^m\sqrt{\pi}}}e^{\frac{1}{2}z^2}\frac{d^m}{dz^m}e^{-z^2}. \tag{3.20}$$

Then we have an expansion,

$$A_\mu(x^i, x^4) = \sum_{m=0}^{\infty}\mathcal{A}_\mu^{(m)}(x^i)\psi_m(x^4), \tag{3.21}$$

where $\mathcal{A}_\mu^{(m)}(x^i)$ are expansion coefficients, which will be determined later. Next, we perform the gauge transformation by which the component A_4 is set to be zero. By this gauge transformation, the components of the gauge field A_i is transformed as

$$A_i \longrightarrow \hat{g}A_i\hat{g}^{-1} + \hat{g}\partial_i\hat{g}^{-1}, \tag{3.22}$$

where the gauge parameter $\hat{g}$ is given by

$$\hat{g}(x^i, x^4) = -\mathcal{P}\exp\left[\int_{-\infty}^{x^4} d\xi\, A_4(x^i,\xi)\right]. \tag{3.23}$$

[4] We sometimes call this the three-dimensional action in Euclid space.

[5] Note that this definition of the Hermite function differs the weight from the usually definition, thus a orthogonal condition of this definition becomes $\int_{-\infty}^{\infty} dz\, \psi_m(z)\psi_n(z) = \delta_{mn}$.

Here the symbol $\mathcal{P}$ stands for the path-ordering. The asymptotic behavior of the Hermite function $\psi_m(\infty) = 0$ and the boundary condition $A_i(x^i, \infty) = 0$ determines the gauge field $A_i(x^i, x^4)$ in the gauge $A_4 = 0$. This is given by [50],

$$A_i(x^i, x^4) = u_i(x^i)\psi_+(x^4) + \sum_{m=0}^{\infty} W_i^m(x^i)\psi_m(x^4), \tag{3.24}$$

where $\psi_+(z) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(z/\sqrt{2})$ and the error function is defined by $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z d\xi\, e^{-\xi^2}$. The gauge field is decomposed into the "zero-mode" $u(x^i)$:

$$u_i(x^i) = U\partial_i U^{-1}, \qquad U(x^i) = \hat{g}(x^i, x^4 = \infty), \tag{3.25}$$

and the infinite tower of the vector fields $W_i^m(x^i)$. The zero-mode $u(x^i)$ is identified with the Skyrme field while the higher modes $W_i^n(x^i)$ can be interpreted as "vector mesons". This analysis is completely parallel to the Kaluza-Klein reduction in which a field is expanded by the Fourier modes $e^{imx^4/2\pi R}$ along the compact circle $x^4 \sim x^4 + 2\pi R$. Note that the expansion along an infinite line enable us to realize the Skyrme field U by the holonomy of the gauge field:

$$U(x^i) = -\mathcal{P}\exp\left[\int_{-\infty}^{\infty} dx^4\, A_4(x^i, x^4)\right]. \tag{3.26}$$

Although it is possible to compute W_i^m, let us focus on the leading approximation, *i.e.* we neglect all the vector meson modes and focus only on the Skyrme field $U(x^i)$. We call this the Sutcliffe's truncation. Then, in the gauge $A_4 = 0$, we have the following decomposition of the gauge field strength:

$$\begin{aligned} F_{i4} &= U\partial_i U^{-1}\partial_4\psi_+(x^4) = R_i\frac{\psi_0(x^4)}{\sqrt{2}\pi^{\frac{1}{4}}}, \\ F_{ij} &= [R_i, R_j]\psi_+(x^4)(\psi_+(x^4) - 1), \end{aligned} \tag{3.27}$$

where $R_i = U\partial_i U^{-1}$ is interpreted as the right current.

Now it is easy to show that the Sutcliffe's truncation of the Yang-Mills action (3.19) gives the energy functional for the static Skyrme field. Plugging the decomposition (3.27) into the quadratic Yang-Mills action (3.19) and performing the integration over x^4, then we find

$$S = \frac{1}{\kappa g^2}\int d^3x\left(-\frac{c_1}{2}\mathrm{Tr}[R_iR_i] - \frac{c_2}{16}\mathrm{Tr}[R_i, R_j]^2\right), \tag{3.28}$$

where the numerical factors are calculated as $c_1 = \frac{1}{4\sqrt{\pi}} \simeq 0.141$, $c_2 = 2\int_{-\infty}^{\infty} dx^4\psi_+^2(\psi_+ - 1)^2 \simeq 0.198$. These numerical factors can be set to $c_1 = c_2 = 1$ by the rescalings of the length $x^i \to \sqrt{c_2/c_1}x^i$ and the overall factor of the action $S \to \frac{1}{\sqrt{c_1c_2}}S$. We therefore consider the natural unit $c_1 = c_2 = 1$ and set $\kappa = 1, g = 1$ for simplicity. After the rescaling, the action (3.28) becomes the energy functional for the static Skyrme field:

We note that the energy functional (3.4) breaks the scale invariance presented in the Yang-Mills action. A physical origin of this violation comes from the Sutcliffe's truncation (3.27) where only the zero-mode (Skyrme field) is taken into account. Once we include all the vector meson modes W_i^n, the scale invariance is expected to be recovered.

3.3 Atiyah-Manton construction

Although the analytic solutions of Skyrmions have not been found so far, there are two methods of constructing approximate charge B Skyrmions with using ansatz, known as a product ansatz [8, 55, 56] and a rational map ansatz [46]. However both of these approximantions have disadvantages. The problem with the product ansatz is that it is only a good description of each unit charge Skyrmion well separated. The rational map ansatz suffers from the opposite deficiency, in that it provides a good approximation to Skyrmions of minimal energy, and also to some low energy saddle point solutions, but does not contain any degrees of freedom without the center position to allow the individual Skyrmions to separate. In this section we describe another method known as an Atiyah-Manton construction. The Atiyah-Manton construction produce good approximations to the minimal energy solutions which including well separated Skyrmions with arbitrary positions and orientation. This approach is based on existence of Yang-Mills instantons, thus if there are not the analytic solutions of instantons we can not employ this approach.

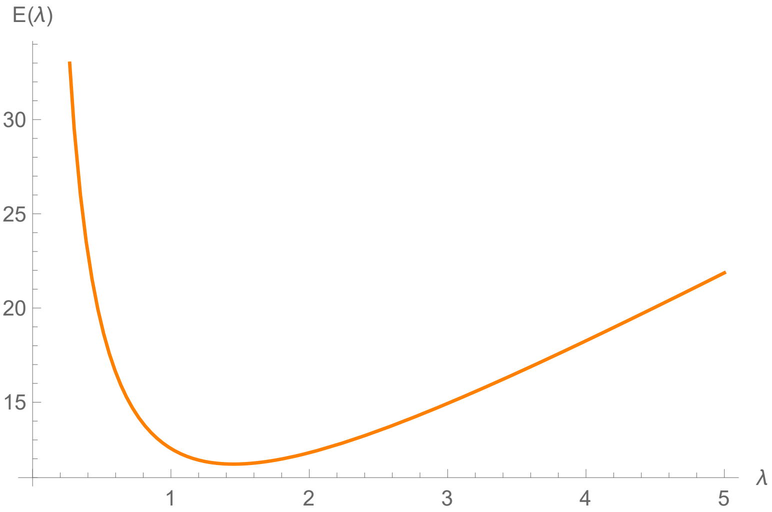


Figure 3.2: The energy profile for the Atiyah-Manton solution (3.32) as the function of the instanton size λ.

Following Atiyah and Manton [10], we calculate the holonomy for the instanton solution. For (1.20), the 't Hooft one-instanton is gien by

$$A_\mu = \frac{1}{4}\partial_\nu \ln\left(1 + \frac{\lambda^2}{||\tilde{x}||^2}\right)\eta^{(-)}_{\mu\nu}, \tag{3.29}$$

where $||\tilde{x}||^2 = (x^\mu - a^\mu)(x_\mu - a_\mu)$ and λ, a^μ are the size and the position moduli of the solution. For simplify, we set $a^\mu = 0$. The symbol $\eta^{(-)}_{\mu\nu}$ is the 't Hooft symbol which is defined by (1.11). To this end, it is convenient to rewrite the solution (3.29) as

$$A_\mu(x^i, x^4) = \frac{1}{2}\left(\frac{1}{\lambda^2 + r^2 + (x^4)^2} - \frac{1}{r^2 + (x^4)^2}\right)x^\nu \eta^{(-)}_{\mu\nu}. \tag{3.30}$$

Then one finds

$$A_4 = \left(\frac{1}{\lambda^2 + r^2 + (x^4)^2} - \frac{1}{r^2 + (x^4)^2}\right)x^i e^\dagger_i. \tag{3.31}$$

Using this representation, we calculate the following holonomy for the one-instanton solution A_4:

$$U(x^i) = -\mathcal{P}\exp\int_{-\infty}^{\infty} dx^4\, A_4(x^i, x^4) = \exp\left[\pi\left(1 - \frac{r}{\sqrt{r^2 + \lambda^2}}\right)\hat{x}^i e^\dagger_i\right]. \tag{3.32}$$

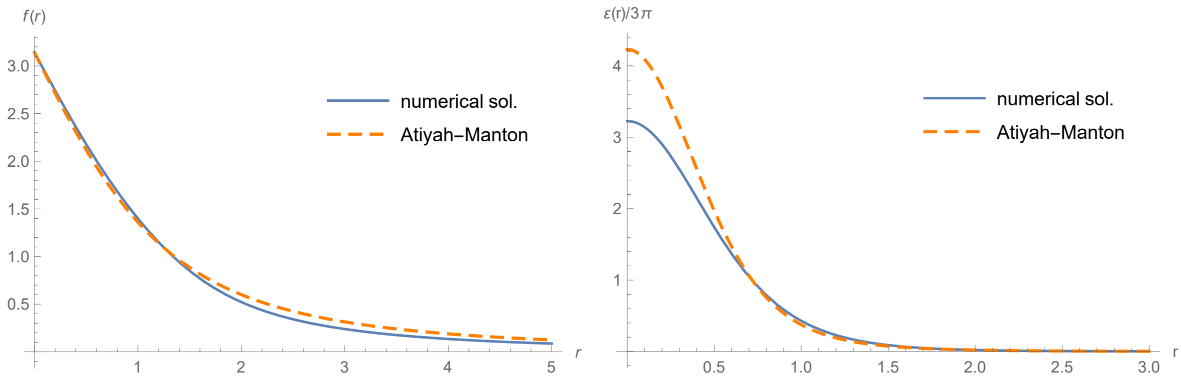
The result is the hedgehog form for the Skyrme field with a profile function given by

$$f(r) = \pi\left(1 - \frac{r}{\sqrt{r^2 + \lambda^2}}\right). \tag{3.33}$$

Instantons are scale invariant, so the (instanton size) parameter λ is arbitrary and we employ to the value which minimize the energy resulting Skyrme field. Hence we have to seek the minimum $E(\lambda)$, $E(\lambda)$ denote the energy functional that the profile function of the Atiyah-Manton solution (3.33) plugging into (3.9). The plot for $E(\lambda)$ is Fig 3.2. We find the minimum point of $E(\lambda)$ at $\lambda = 1.45227$.

For this value of λ, we now compare the profile functions of the Atiyah-Manton and the solution by numerical analysis (see Fig 3.3(a)). The plot for the energy density is also compared in Fig 3.3. We find that they agree with good accuracy. This result can be confirmed by evaluating the total energy (see Table 3.1). Therefore the Atiayh-Manton solutions is the good approximations to the numerical solutions.

In higher charges, such that when we start from the more general instantons, the Atiyah-Manton construction also give the good approximate solutions of Skyrmions, but the calculation of the holonomy can not be perfomed analytically, it can be done numerically only. For further details of the higher charges case see [57, 58, 59].



(a) The profile functions plot.

(b) The energy densities plot.

Figure 3.3: The numerical versus the Atiyah-Manton solutions.

Solution	Numerical sol.	Atiyah-Manton	BPS bound
Energy	$1.2314 \times 12\pi^2$	$1.2432 \times 12\pi^2$	$12\pi^2$

Table 3.1: The total energy for the numerical, the Atiyah-Manton solutions and the BPS bound in the Skyrme model (3.4).

Chapter 4

Skyrme models in eight dimensions and more higher dimensions

In this chapter we introduce a Skyrme model in eight dimensions following the formalism developed by Sutcliffe [50]. Furthermore we discuss the spherically symmetric Skyrmion and the Atiyah-Manton construction in eight dimensions. The discussion of this chapter is based on [60].

The instantons in four dimensions satisfy the self-duality equation $F = *_4 F$. Here F is the field strength 2-form of the gauge field and $*_d$ is the Hodge dual operator in d dimensions. A natural higher-dimensional generalization of instantons is a solution to the self-duality equations in $d = 4n$ dimensions $F(n) = *_{4n} F(n)$ where $F(n)$ is the n wedge products of F. The $n = 1$ case corresponds to the instantons in four-dimensions while the $n \geq 2$ cases are their generalization. The first non-trivial example is the $n = 2$ case, namely, the self-dual instantons in eight dimensions. This was studied so far from various contexts [30, 31]. Futhermore the higher dimensional instantons were discussed in chapter 2. On the other hand, it is possible to consider higher-dimensional generalizations of Skyrmions [61].

In this chapter we study the relation between instantons and Skyrmions in higher dimensions. In particular, we focus on the eight-dimensional self-dual instantons that satisfy $F \wedge F = *_8 F \wedge F$. The self-duality relation is obtained by the Bogomol'nyi completion of the generalized Yang-Mills action in eight dimensions. We will derive the energy functional for the static Skyrme field from the generalized Yang-Mills action by the reduction procedure developed by Sutcliffe [50]. The Derrick's theorem indicates that the model admits static soliton solutions which we call the eight-dimensional Skyrmions. We will find the numerical solution of the above mentioned Skyrmion. We will then calculate a field configuration through the Atiyah-Manton construction applied to the eight-dimensional instanton and find that this gives a good approximation to the numerical solution of the Skyrmion. These results strongly suggest that the instanton/Skyrmion correspondence holds even in $4n$ dimensions and this relation is an universal property.

The organaization of this chapter as follows. Section 4.1 is about an eight-dimensional Skyrme model. In this section, we lead to the Skyrme model in eight dimensions from the eight-dimensional generalized Yang-Mills model with using the Sutcliffe's truncation method. Section 4.2 is discussion of the eight-dimensional single Skyrmions from leading by two methods. First method is that we directly solve an equation of motion with a spherically symmetric ansatz numerically. The other method is construcion that single Skyrmion from a holonomy of the eight-dimensional 't Hooft 1-instanton, namely a higher-dimensional Atiyah-Manton construction. We will show that the Atiyah-Manton construcion in higher dimensions works well also. We introduce a seven-dimensional hedgehog ansatz as spherically symmetric ansatz. Section 4.3 is discussed an $4n$-dimensional Skyrme model from the $4n$-dimensional generalized Yang-Mills model. In this section, we will show that the $4n$-dimensional Skyrme model satisfy the Derrick's theorem, thus it can be expected that the existence of solitonic solutions, namely higher-dimensional Skyrmions, in this model. Futhermore we lead an explicit action(energy functional) of twelve-dimensional Skyrme model from the generalized Yang-Mills model in twelve dimensions. In section 4,4, we show the some calculations in more detail. Specifically as follows. In subsection 4.4.1, we show the Bogomol'nyi completion of the eight-dimensional (static) Skyrme model, Moreover we lead the normalization constant. Subsection 4.4.2 is about an higher-dimensional sperically symmetric ansatz, namelly an hegehog ansatz in dimensions higher than theree. We introduce the higher dimensional Hedgehog ansatz by using the 4n-dimensional ASD basis, and lead the eight-dimensional Skyrme model with this hegehog ansatz. We find that this hegehog ansatz works well in section 4.2. Section 4.5 is about the numerical calculation to calculate the single Skyrmion in eight dimensions.

4.1 Eight-dimensional Skyrme model

Now we generalize the procedure in the previous chapter to eight dimensions. In eight dimensions, the natural action whose BPS equation is the self-duality equation $F \wedge F = *_8(F \wedge F)$ is that of the generalized Yang-Mills action. The action is

$$\begin{aligned} S_{\text{gYM}} &= \frac{\alpha}{\kappa g^2} \int \text{Tr}\,[*_8(F \wedge F) \wedge (F \wedge F)] \\ &= \left(\frac{1}{2!}\right)^4 \frac{4}{4!} \frac{\alpha}{\kappa g^2} \int d^8x \, \text{Tr}\left[(F^{\mu\nu}F^{\rho\sigma})^2 - 4F^{\mu\nu}F^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + (F^{\mu\nu}F_{\mu\nu})^2\right]. \end{aligned} \tag{4.1}$$

Here $\mu, \nu, \ldots = 1, \ldots, 8$ and the component of the gauge field strength 2-form $F = \frac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu$ is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. A constant α has mass dimension $[\alpha] = -4$ and g is the gauge coupling constant whose mass dimension is -2. In the following, we set $\alpha/g^2 = 96$ and $\kappa = 1$ for simplicity. The gauge field A_μ is in the adjoint representation of a Lie algebra associated with the gauge group G. We consider a gauge group G which admits a non-trivial homotopy $\pi_7(G) = \mathbb{Z}$. The analysis is completely parallel to the four-dimensional case. We decompose the directions $x^\mu = (x^i, x^8)$, $(i = 1, \ldots, 7)$ and expand the gauge field in terms of the Hermite function $\psi_m(x^8)$. The Sutcliffe's truncation provides the static Skyrme field in eight dimensions through the relations (3.27). Plugging the expansion (3.27) into the generalized Yang-Mills action (4.1) and performing the integration over the x^8-direction, we obtain the energy functional for the static Skyrme field. Let us show this calculation as follows. The first term in (4.1) becomes

$$\text{Tr}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^2\right] = \text{Tr}\left[\left(F^{ij}F_{ij}\right)^2 + 4F^{ij}F_{ij}F^{k8}F_{k8} + 4\left(F^{i8}F_{i8}\right)^2\right]. \tag{4.2a}$$

The second term in (4.1) becomes

$$\text{Tr}\left[(F^{\mu\nu}F^{\rho\sigma})^2\right] = \text{Tr}\left[\left(F^{ij}F^{kl}\right)^2 + 4\left(F^{ij}F^{k8}\right)^2 + 4\left(F^{i8}F^{j8}\right)^2\right]. \tag{4.2b}$$

The third term in (4.1) becomes

$$\text{Tr}\left[F^{\mu\nu}F^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right] = \text{Tr}\left[F^{ij}F^{kl}F_{ik}F_{jl} + F^{ij}F^{k8}F_{ik}F_{j8} - 2F_{ij}F^{ki}F^{j8}F_{k8} + F^{ij}F_{i8}F_{kj}F^{k8} + \left(F^{i8}F^{j8}\right)^2 + \left(F^{i8}F_{i8}\right)^2\right]. \tag{4.2c}$$

Hence

$$\begin{aligned} &\text{Tr}\left[\left(F^{\mu\nu}F_{\mu\nu}\right)^2 + (F^{\mu\nu}F^{\rho\sigma})^2 - 4\left(F^{\mu\nu}F^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right)\right] \\ &= \text{Tr}\Big[\left(F^{ij}F_{ij}\right)^2 + \left(F^{ij}F^{kl}\right)^2 - 4F^{ij}F^{kl}F_{ik}F_{jl} \\ &\qquad + 4F^{ij}F_{ij}F^{k8}F_{k8} + 4\left(F^{ij}F^{k8}\right)^2 - 4F^{ij}F^{k8}F_{ik}F_{j8} + 8F_{ij}F^{kl}F^{j8}F_{k8} - 4F^{ij}F_{i8}F_{kj}F^{k8}\Big]. \end{aligned} \tag{4.3}$$

In eight dimensions, the decomposition of the gauge field strength with Sutcliffe's truncation (3.27) becomes

$$F_{i8}(\mathbf{x}, x^8) = R_i(\mathbf{x}) \frac{\psi_0(x^8)}{\sqrt{2}\pi^{1/4}}, \tag{4.4a}$$

$$F_{ij}(\mathbf{x}, x^8) = [R_i(\mathbf{x}), R_j(\mathbf{x})]\psi_+(x^8)(\psi_+(x^8) - 1). \tag{4.4b}$$

Therefore we obtaitn the (static) Skyrme model in eight dimension:

$$\begin{aligned} E_{\text{Skyrme}} = \int d^7x \text{Tr}\Big[&c_2\left([R_i, R_j][R_i, R_j]\right)^2 + c_2\left([R_i, R_j][R_k, R_l]\right)^2 \\ &- 4c_2[R_i, R_j][R_k, R_l][R_i, R_k][R_j, R_l] \\ &+ 4c_1\left([R_i, R_j]\right)^2 R_k^2 + 4c_1\left([R_i, R_j]R_k\right)^2 - 4c_1[R_i, R_j]R_k[R_i, R_k]R_j \\ &+ 8c_1[R_i, R_j][R_k, R_i]R_jR_k - 4c_1[R_i, R_j]R_i[R_k, R_j]R_k\Big]. \end{aligned} \tag{4.5}$$

Here $R_i = U\partial_i U^{-1}$ is the right current and the Skyrme field is defined by the holonomy

$$U(x^i) = -\mathcal{P}\exp\left[\int_{-\infty}^{\infty} dx^8 \, A_8(x^i, x^8)\right]. \tag{4.6}$$

Therefore, the Skyrme field is a map $U : \mathbb{R}^7 \mapsto \tilde{G}$ where $\tilde{G}$ is a group manifold. The numerical constants c_1, c_2 in (4.5) are calculated to be

$$c_1 = \int_{-\infty}^{\infty} dx^8 \ \frac{1}{2\sqrt{\pi}} \psi_0^2 \psi_+^2 (\psi_+ - 1)^2 \simeq 0.00940, \quad c_2 = \int_{-\infty}^{\infty} dx^8 \ \psi_+^4 (\psi_+ - 1)^4 \simeq 0.00308. \tag{4.7}$$

As in the case of the four-dimensional Skyrme model, these numerical factors are scaled away by the replacements $x^i \to \sqrt{c_2/c_1} x^i$, $E_{\text{Skyrme}} \to \frac{1}{\sqrt{c_1 c_2}} E_{\text{Skyrme}}$. We therefore set $c_1 = c_2 = 1$. The generalized Yang-Mills action (4.1) has the scale invariance while the energy functional (4.5) does not. Again, this is due the Sutcliffe's truncation where only the zero-mode is considered and the vector mesons are neglected.

The eight-dimensional Skyrme model (4.5) has similar properties with the four-dimensional ones. For example, the energy functional (4.5) is invariant under the following global transformation

$$U \to O_L U O_R^{-1}, \qquad O_L, O_R \in \tilde{G}. \tag{4.8}$$

This is a generalization of the chiral symmetry in four dimensions. One also finds that the energy functional (4.5) consists of the terms with 6th and 8th derivatives. This is compared with the 2nd and 4th derivative terms in the four-dimensional Skyrme model. The Derrick's theorem applied to the energy (4.5) indicates that there is a stable solitonic solution to this model. We call this the eight-dimensional Skyrmions. The Bogomol'nyi completion of the energy (4.5) is given by

$$E_{\text{Skyrme}} = 4 \int d^7 x \ \mathrm{Tr} \left[\left(\sqrt{\frac{1}{3!}} \varepsilon_{ijklmno} R_i R_j R_k \pm \sqrt{4!} R_{[l} R_m R_n R_{o]} \right)^2 \mp 4 \varepsilon_{ijklmno} R_i R_j R_k R_l R_m R_n R_o \right] \geq \frac{16}{N_C} |\mathcal{B}|, \tag{4.9}$$

where $N_C = -1/9600\pi^4$ is the normalization constant of the following topological charge:

$$\mathcal{B} = N_C \int d^7 x \ \mathrm{Tr} \left[\varepsilon_{ijklmno} R_i R_j R_k R_l R_m R_n R_o \right]. \tag{4.10}$$

Here $\varepsilon_{ijklmno}$ is the totally antisymmetric tensor. The topological charge (4.10) is the natural generalization of the Baryon number $B = \frac{1}{24\pi^2} \int d^3 x \ \mathrm{Tr}[\varepsilon_{ijk} R_i R_j R_k]$ in the four-dimensional Skyrme model. This calculation in more detail see subsection 4.4.1.

4.2 Eight-dimensional Skyrmions from instantons

In this section, we examine a field configuration that extremizes the energy functional (4.5), namely, the Skyrmion in eight dimensions. Assuming the hedgehog ansatz for the Skyrme field $U(x)$, we first derive the equation of motion from (4.5). We will find a solution to the equation by the numerical analysis. We then construct a field configuration from the eight-dimensional instantons through the Atiyah-Manton prescription. We compare the two solutions and verify whether the Atiyah-Manton approximation works even in eight dimensions.

4.2.1 Skyrmions from numerical analysis

Following the standard scheme for a spherically symmetric solution to the four-dimensional Skyrme model, we consider the following hedgehog ansatz:

$$U(x) = \exp \left(f(r) \hat{x}^i e_i^\dagger \right), \tag{4.11}$$

where $\hat{x}^i = \frac{x^i}{r}$, $r^2 = x^i x^i$ and $f(r)$ is a real function. The basis $e_i, e_i^\dagger$ is the higher dimensional analogue of the pure imaginary quaternions in four dimensions. Note that we do not employ the octonions as a higher dimensional generalisation of the quaternions. It is well known that the octonions are never represented by matrices and the algebra based on them loses the associativity [62]. The natural candidate for the basis in eight dimensions is based on the Clifford algebra. This is given by

$$e_\mu = \delta_{\mu 8} \mathbf{1}_8 + \delta_{\mu i l} \Gamma_i^{(-)}, \qquad e_\mu^\dagger = \delta_{\mu 8} \mathbf{1}_8 + \delta_{\mu i} \Gamma_i^{(+)}, \qquad (\mu = 1, \ldots, 8, \ i = 1, \ldots, 7), \tag{4.12}$$

where $\Gamma_i^{(\pm)}$ are 8×8 matrices that satisfy the relations $\{\Gamma_i^{(\pm)}, \Gamma_j^{(\pm)}\} = -2\delta_{ij} \mathbf{1}_8$. The matrices $\Gamma_i^{(\pm)}$ are defined by $\Gamma_i^{(\pm)} = \frac{1}{2}(1 \pm \omega)\Gamma_i$. We choose the matrices $\Gamma_i^{(\pm)}$ such that they satisfy the relation $\Gamma_i^{(+)} = -\Gamma_i^{(-)}$. Here Γ_i are given by the matrix representation of the seven-dimensional complex Clifford algebra $\Gamma_i \in C\ell_7(\mathbb{C})$ and $\omega = (-1)\Gamma_1 \cdots \Gamma_7$ is a chirality matrix. The basis is normalized as $\mathrm{Tr}[e_\mu e_\nu^\dagger] = 8\delta_{\mu\nu}$ and satisfies the following relations

$$
\begin{aligned}
&e_\mu e_\nu^\dagger + e_\nu e_\mu^\dagger = e_\mu^\dagger e_\nu + e_\nu^\dagger e_\mu = 2\delta_{\mu\nu}\mathbf{1}_8, \\
&e_\mu e_\nu + e_\nu e_\mu = 2\delta_{\mu 8} e_\nu + 2\delta_{\nu 8} e_\mu - 2\delta_{\mu\nu}\mathbf{1}_8, \\
&e_\mu^\dagger e_\nu^\dagger + e_\nu^\dagger e_\mu^\dagger = 2\delta_{\mu 8} e_\nu^\dagger + 2\delta_{\nu 8} e_\mu^\dagger - 2\delta_{\mu\nu}\mathbf{1}_8.
\end{aligned} \tag{4.13}
$$

Note that we have $e_i^\dagger = -e_i$ in our construction. Therefore the hedgehog field configuration (4.11) satisfies $U^\dagger U = \mathbf{1}_8$ and it belongs to $U(8)$. The details of the Clifford algebra, including the explicit matrix representations of the basis $e_\mu, e_\mu^\dagger$, are found in section 2.5.

We now derive the equation of motion for the profile function $f(r)$ (The following calculations in more detail see subsection 4.4.2). Using the algebra of the basis (4.13), we find that the hedgehog ansatz is expanded as

$$U(x) = \cos f \mathbf{1}_8 + \sin f \hat{x}^i e_i^\dagger. \tag{4.14}$$

This expression allows us to write down the right-current field:

$$R_i = r^{-1}\sin^2 f \hat{x}_i \mathbf{1}_8 - (-r^{-1}\sin f\cos f + \partial_r f)\hat{x}_i \hat{x}^\dagger - r^{-1}\sin f\cos f e_i^\dagger + r^{-1}\sin^2 f e_i^\dagger \hat{x}^\dagger. \tag{4.15}$$

Here $\hat{x} = \hat{x}^i e_i$, $\hat{x}^\dagger = \hat{x}^i e_i^\dagger$. It is straightforward to calculate each term in (4.5) by using the above expression and the algebra associated with the basis (4.13). The energy functional becomes,

$$
\begin{aligned}
E_{\text{Skyrme}} &= \int_0^\infty dr \int_{S^6} d\Omega_6 \mathcal{E}(r) \\
&= 24576\pi^3 \int_0^\infty dr \left(3r^2 \sin^4 f (\partial_r f)^2 + 4\sin^6 f(4(\partial_r f)^2 + 1) + 12\frac{\sin^8 f}{r^2}\right),
\end{aligned} \tag{4.16}
$$

where the overall factor comes from the volume factor of the radial direction and algebras containing $e_i, e_i^\dagger$. Then, we derive the equation of motion for $f(r)$ as

$$
\begin{aligned}
&\sin^2 f(3r^2 + 16\sin^2 f)\partial_r^2 f + 6r\sin^2 f\partial_r f \\
&\quad + 3\sin 2f\left[(r^2 + 8\sin^2 f)(\partial_r f)^2 - 2\sin^2 f - 8\frac{\sin^4 f}{r^2}\right] = 0.
\end{aligned} \tag{4.17}
$$

The boundary condition for the profile function $f(r)$ is

$$f(0) = \pi, \qquad f(\infty) = 0. \tag{4.18}$$

Compared with the equation in four dimensions, the equation (4.17) looks highly non-linear. Therefore it is not obvious whether the equation (4.17) has appropriate solutions that are consistent with the boundary condition (4.18) or not. In order to clarify the existence of the solution to the equation (4.17), we first perform the Taylor expansion of the profile function at the origin: $f(\delta r) = \sum_{i=0}^\infty f_i (\delta r)^i = f_0 + f_1\delta r + f_2(\delta r)^2 + \ldots$, namely we perform analyzation that similar to four dimensional case. We then write down the equations for the coefficients f_i and look for f_i order by order in $(\delta r)^i$. For the boundary condition (4.18), we find that the asymptotic behavior of the solution around the origin is

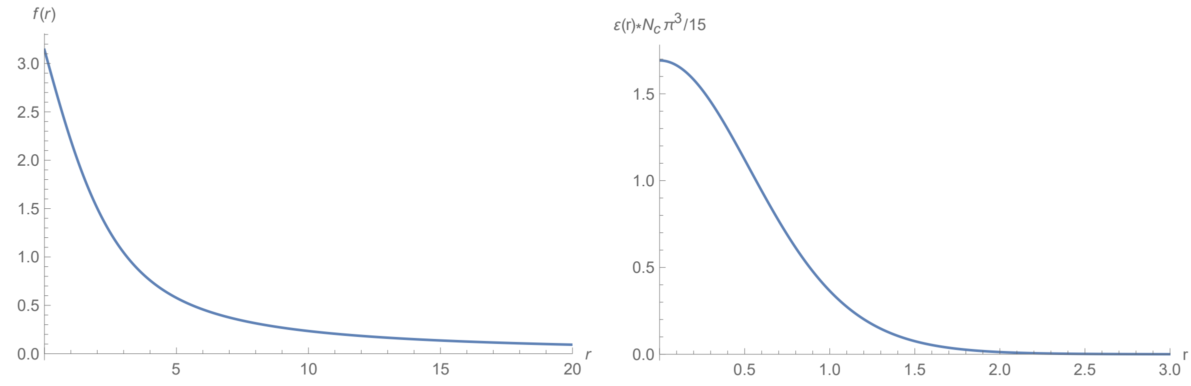
$$f(\delta r) = \pi + f_1\delta r - \frac{\left(3 + 8f_1^2\right)f_1^3}{9\left(3 + 16f_1^2\right)}(\delta r)^3 + \frac{2f_1^5\left(387 + 16f_1^2\left(192 + 789f_1^2 + 1616f_1^4\right)\right)}{1485\left(3 + 16f_1^2\right)^3}(\delta r)^5 + O\left((\delta r)^7\right). \tag{4.19}$$

Here f_1 can be chosen as a shooting parameter in the numerical analysis. From this observation, we conclude that we can numerically calculate a solution to the equation (4.17) by appropriate methods of second ordinary differential equations with boundary conditions. Moreover we find that the seven-dimensional Hedgehog ansatz (4.11) is proper spherically symmetric ansatz. The numerical result is found in Fig. 4.1 where we have employed the functional Newton-Raphson method. The behaviour of the profile function and the energy functional is quite similar to those in the four-dimensional Skyrmion (see Fig. 3.1). A specific method for this numerical analysis in more detail see below section 4.5.

The Skyrme field is a map $\mathbb{R}^7 \mapsto \mathrm{U}(8)$. However, the boundary condition $U(r) \to \mathbf{1}_8$ ($r \to \infty$) implies that the base manifold is topologically S^7. Therefore the solutions are characterized by the topological charge associated with the homotopy group $\pi_7(\mathrm{U}(8)) = \mathbb{Z}$. Indeed, the topological charge for the hedgehog ansatz (4.11) and the boundary condition (4.18) is evaluated to be

$$\mathcal{B} = 9600\pi^3 N_C(f(\infty) - f(0)) = 1. \tag{4.20}$$

This is the single Skyrmion in eight dimensions.



(a) The profile for the 8d Skyrmion.

(b) The energy density plot for the 8d Skyrmion.

Figure 4.1: The numerical profile for $f(r)$ and the plot for the energy density $\mathcal{E}(r)$.

4.2.2 Atiyah-Manton solution from instantons

We next make contact with the Skyrmion from the eight-dimensional instantons. The Bogomol'nyi completion of the generalized Yang-Mills action (4.1) is

$$\begin{aligned} S_{\text{gYM}} &= \frac{\alpha}{2\kappa g^2} \int \mathrm{Tr}\left[(F \wedge F \mp *_8 (F \wedge F))^2 \pm 2F \wedge F \wedge F \wedge F \right] \\ &\geq \pm \frac{\alpha}{\kappa g^2} \int \mathrm{Tr}[F \wedge F \wedge F \wedge F]. \end{aligned} \tag{4.21}$$

Here we have defined

$$(F \wedge F \pm *_8 F \wedge F)^2 = (F \wedge F \pm *_8 F \wedge F) \wedge *_8 (F \wedge F \pm *_8 F \wedge F). \tag{4.22}$$

The action is bounded from below by the fourth Chern number $k = \int \mathrm{Tr}[F \wedge F \wedge F \wedge F]$ which defines the topological charge associated with instantons. The theory defined by the action (4.21) has scale invariance. The Derrick's theorem implies that the theory admits static solitons, namely, instantons. The Bogomol'nyi bound is saturated when the (anti-)self-duality equation

$$F \wedge F = \pm *_8 F \wedge F, \tag{4.23}$$

is satisfied. This is a natural generalization of the (anti-)self-duality equation $F = \pm *_4 F$ in four dimensions. In the following we choose the plus sign in (4.23). Solutions to the equation (4.23) is known as the self-dual instantons in eight dimensions. They are characterized by the homotopy group $\pi_7(G) = \mathbb{Z}$ where G is a gauge group. The 't Hooft type one-instanton solution is given by

$$A_\mu = \frac{1}{4} \partial_\nu \ln \left(1 + \frac{\lambda^2}{||\tilde{x}||^2} \right) \Sigma^{(-)}_{\mu\nu}, \tag{4.24}$$

where $||\tilde{x}||^2 = (x^\mu - a^\mu)(x_\mu - a_\mu)$ and λ, a^μ are the size and the position moduli of the solution. For simplify, we set $a^\mu = 0$. The matrix

$$\Sigma^{(-)}_{\mu\nu} = e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger \tag{4.25}$$

is the ASD tensor which is the eight-dimensional analogue of the 't Hooft instanton in four dimensions (see previous chapter 2).

Similar as the four-dimensional case, we calculate the holonomy for the instanton solution (4.24). To this end, it is convenient to rewrite the solution (4.24) as

$$A_\mu(x^i, x^8) = \frac{1}{2} \left(\frac{1}{\lambda^2 + r^2 + (x^8)^2} - \frac{1}{r^2 + (x^8)^2} \right) x^\nu \Sigma^{(-)}_{\mu\nu}. \tag{4.26}$$

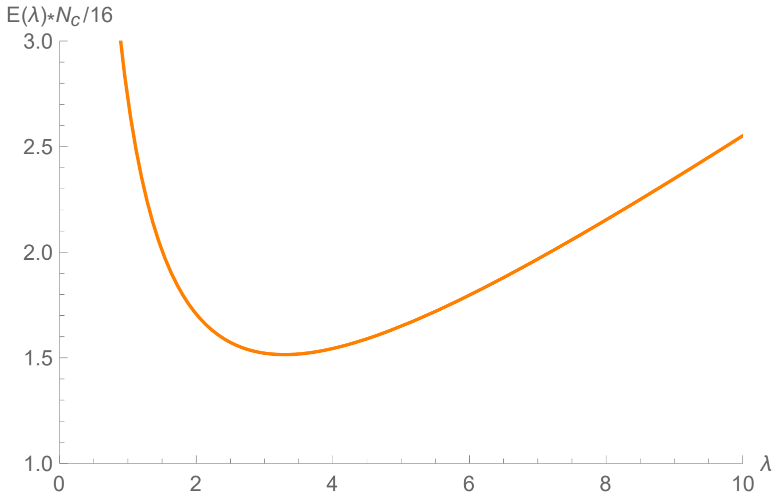


Figure 4.2: The energy profile for the Atiyah-Manton solution (4.28) as the function of the instanton size λ.

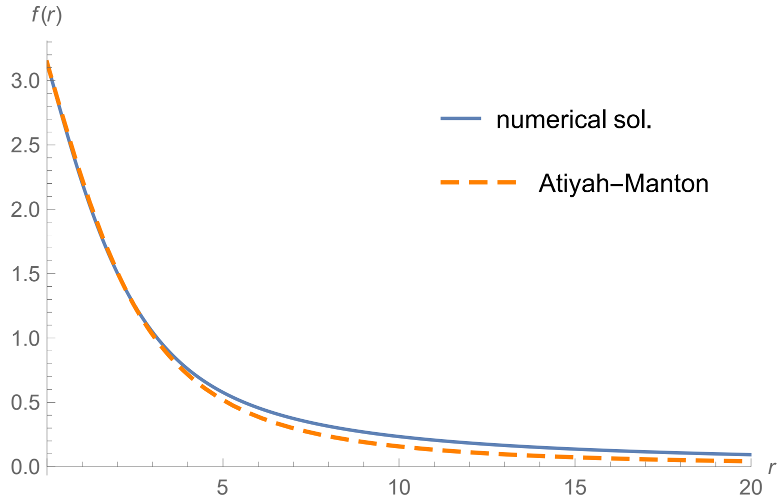


Figure 4.3: The profile functions for the numerical and the Atiyah-Manton solutions.

Then one finds

$$A_8 = \left(\frac{1}{\lambda^2 + r^2 + (x^8)^2} - \frac{1}{r^2 + (x^8)^2} \right) x^i e_i^\dagger. \tag{4.27}$$

Using this representation, we calculate the following holonomy for the one-instanton solution A_8:

$$U(x^i) = -\mathcal{P} \exp \int_{-\infty}^{\infty} dx^8 \, A_8(x^i, x^8) = \exp \left[\pi \left(1 - \frac{r}{\sqrt{r^2 + \lambda^2}} \right) \hat{x}^i e_i^\dagger \right]. \tag{4.28}$$

The result is the standard hedgehog form for the Skyrme field (4.11). This is why we have employed the basis $e_i^\dagger$ in (4.11). Plugging the Atiyah-Manton solution (4.28) into the (static) Skyrme action with the hedgehog ansatz (4.16) results in the static energy $E(\lambda)$ for the solution. The plot for $E(\lambda)$ is found in Fig. 4.2. As anticipated, the energy depends on the size of the instanton λ. This is because the Sutcliffe's truncation breaks the scale invariance in the generalized Yang-Mills model. The size λ now lost its status of modulus. The true solution corresponds to the extremum of $E(\lambda)$. We find this happens at $\lambda = 3.29095$.

For this value of λ, we now compare the profile functions of the Atiyah-Manton with the numerical solutions. The result is found in Fig 4.3. We find that they agrees with high accuracy. The plot for the energy density is also compared in Fig 4.4. Again, we find a good agreement between them. This result can be confirmed by evaluating the total energy (see Table 4.1). We

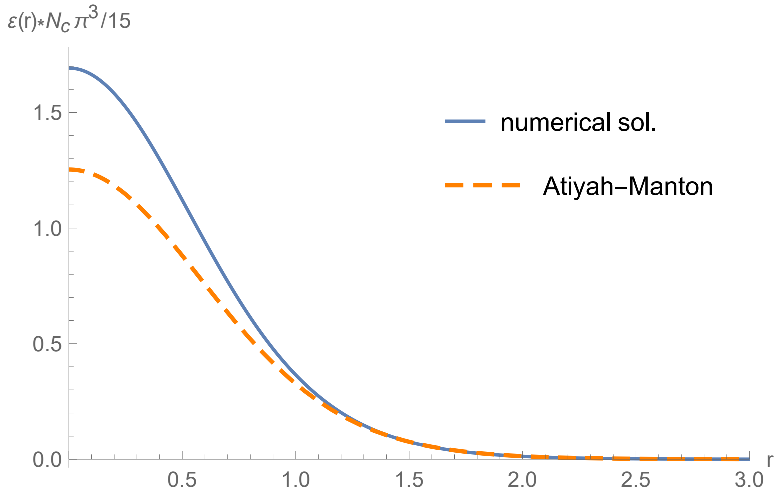


Figure 4.4: The profile functions for the energy density. The numerical versus the Atiyah-Manton solutions.

Solution	Numerical	Atiyah-Manton	BPS bound
Energy	$1.51239 \times 16/N_c$	$1.51521 \times 16/N_c$	$16/N_c$

Table 4.1: The total energy for the numerical, the Atiyah-Manton solutions and the BPS bound in this model (4.9).

therefore conclude that the Atiyah-Manton construction of Skyrmions from instantons works well even in eight dimensions. We note that the eight-dimensional Skyrmion is a non-BPS solution which is same as the four-dimensional one.

4.3 The higher dimensional generalization

In this section we perform an analysis on the Sutcliffe's truncation method in $4n$ dimensions. It is worthwhile first to mention about the $n = 3$, namely, the twelve-dimensional case. In twelve dimensions, the self-duality equation becomes $F \wedge F \wedge F = \pm *_{12} F \wedge F \wedge F$. It is an easy exercise to show that the one-instanton solution to this equation is given by (4.24) where the SO(8) generator $\Sigma^{(-)}_{\mu\nu}$ is replaced by that of SO(12). We can construct the Atiyah-Manton solution by calculating the holonomy associated with the instanton solution. We can also find the Skyrme model in twelve dimensions and its Skyrmion solution along the lines of the eight-dimensional case. The discussion is parallel to that in eight dimensions presented in previous discussion. However, the explicit calculation of the Sutcliffe's truncation in twelve dimensions results in the energy functional for the Skyrme model with diverse (about $\mathcal{O}(10^2)$) terms. Analyzing all the terms is very hard, thus we first proceed to the general discussion in the following.

Now we move to the discussion in $4n$ dimensions. The $4n$-dimensional generalization of the generalized Yang-Mills action (4.1) is

$$S_{\mathrm{YM}} = \int_{\mathbb{R}^{4n}} \mathrm{Tr}[F(n) \wedge *_{4n} F(n)], \tag{4.29}$$

where $F(n)$ is the nth wedge products of the gauge field strength 2-form, $F(n) = F \wedge \cdots \wedge F$. The gauge field takes value in the adjoint representation of a gauge group G. We assume that this gauge group has non-trivial homotopy $\pi_{4n-1}(G) = \mathbb{Z}$. It is straightforward to perform the Bogomol'nyi completion of the action:

$$S_{\mathrm{YM}} = \frac{1}{2} \int_{\mathbb{R}^{4n}} \mathrm{Tr}[(F(n) \mp *_{4n} F(n))^2 \pm 2F(2n)] \geq \pm \int_{\mathbb{R}^{4n}} \mathrm{Tr}[F(2n)]. \tag{4.30}$$

The BPS equation becomes

$$F(n) = \pm *_{4n} F(n). \tag{4.31}$$

This is the (anti-)self-duality equation in $4n$ dimensions. The one-instanton solution to this equation is explicitly wrote down by the ADHM construction of instantons in $4n$ dimensions [26] which is the $4n$-dimensional generalization of [25] in eight dimensions. Again, the solutions are given as the form in (4.24) where the SO(8) generator is replaced by those of SO($4n$).

Next we perform the Sutcliffe's truncation. The index structure of the Yang-Mills Lagrangian is

$$\begin{aligned}&F(n)\wedge *_{4n}F(n)\\&=\frac{1}{(2n)!}\left(\frac{1}{2!}\right)^{2n}\varepsilon_{\mu_1\cdots\mu_{2n}\nu_1\cdots\nu_{2n}}\varepsilon^{\mu_1\cdots\mu_{2n}\rho_1\cdots\rho_{2n}}F^{\nu_1\nu_2}\cdots F^{\nu_{2n-1}\nu_{2n}}F_{\rho_1\rho_2}\cdots F_{\rho_{2n-1}\rho_{2n}}d^{4n}x,\end{aligned}\tag{4.32}$$

where the overall factor comes from the normalization of the 2-form $F=\frac{1}{2!}F_{\mu\nu}dx^\mu\wedge dx^\nu$ and the definition of the Hodge dual operation. The procedure of the reduction is parallel to the previous sections. We can reduce the gauge field along, say, the x^{4n}-direction. Then, the gauge field becomes

$$\begin{aligned}&F_{i\sharp}=R_i\frac{\psi_0(x^{4n})}{\sqrt{2}\pi^{\frac{1}{4}}},\quad F_{ij}=[R_i,R_j]\psi_+(x^{4n})(\psi_+(x^{4n})-1),\\&(i,j,\ldots=1,\ldots 4n-1,\ \sharp=4n).\end{aligned}\tag{4.33}$$

Here $R_i=U\partial_iU^\dagger$ is the right current field constructed from the Skyrme field $U(x^i)$. Then, the energy functional for the static field $U(x^i)$ in $4n$ dimensions has the following structure

$$E_{\text{Skyrme}}=E_{4n}(x)+E_{4n-2}(x),\tag{4.34}$$

where E_m stands for terms that contain m-th derivatives. The energy (4.34) is compared with that in the eight-dimensional Skyrme model. Again, the Derrick's theorem implies that there is a static soliton solution that extremizes the energy (4.34). This is nothing but the Skyrmion in $4n$ dimensions. Finding the explicit solutions need the numerical analysis in each dimension. We can also calculate the holonomy for the 1-instanton solution in $4n$ dimensions and derive the static energy $E(\lambda)$. Although we do not repeat the same calculations, the result of the original Atiyah-Manton construction in four dimensions and our result in eight dimensions strongly suggest that this instanton/Skyrmion correspondence does hold in $4n$ dimensions.

4.3.1 twelve-dimensional Skyrme model

In this subsection, we will lead a twelve-dimensional Skyrme action. For (4.32),

$$(F\wedge F\wedge F)\wedge *_{12}(F\wedge F\wedge F)=\frac{1}{6!}\left(\frac{1}{2!}\right)^6\varepsilon_{\mu_1\ldots\mu_6\nu_1\ldots\nu_6}\varepsilon^{\mu_1\ldots\mu_6\rho_1\ldots\rho_6}F^{\nu_1\nu_2}F^{\nu_3\nu_4}F^{\nu_5\nu_6}F_{\rho_1\rho_2}F_{\rho_3\rho_4}F_{\rho_5\rho_6}d^{12}x,\tag{4.35}$$

thus

$$\begin{aligned}&\frac{1}{6!}\left(\frac{1}{2!}\right)^6\varepsilon_{\mu_1\ldots\mu_6\nu_1\ldots\nu_6}\varepsilon^{\mu_1\ldots\mu_6\rho_1\ldots\rho_6}\mathrm{Tr}\left[F^{\nu_1\nu_2}F^{\nu_3\nu_4}F^{\nu_5\nu_6}F_{\rho_1\rho_2}F_{\rho_3\rho_4}F_{\rho_5\rho_6}\right]\\
&=\frac{1}{8}\mathrm{Tr}\Big[2F^{MN}F_{MN}\left(F^{OP}F^{QR}\right)^2+F^{MN}F_{MN}F^{OP}F^{QR}F_{QR}F_{OP}+\left(F^{MN}F^{OP}F^{QR}\right)^2\\
&\quad+F^{MN}F^{OP}F^{QR}F_{MN}F_{QR}F_{OP}+F^{MN}F^{OP}F^{QR}F_{OP}F_{MN}F_{QR}\\
&\quad-8F^{MN}F_{MN}F^{OP}F^{QR}F_{OQ}F_{PR}-8F^{MN}F^{OP}F^{QR}F_{MN}F_{OQ}F_{PR}-8F^{MN}F^{OP}F^{QR}F_{OP}F_{MQ}F_{NR}\\
&\quad-4F^{MN}F^{OP}F^{QR}F_{MQ}F_{OP}F_{NR}-4F^{MN}F^{OP}F^{QR}F_{MO}F_{QR}F_{NP}-4F^{MN}F^{OP}F^{QR}F_{MQ}F_{NR}F_{OP}\\
&\quad+8F^{MN}F^{OP}F^{QR}F_{MO}F_{NQ}F_{PR}-8F^{MN}F^{OP}F^{QR}F_{MO}F_{PQ}F_{NR}+8F^{MN}F^{OP}F^{QR}F_{OQ}F_{MP}F_{NR}\\
&\quad-8F^{MN}F^{OP}F^{QR}F_{OQ}F_{MR}F_{NP}+8F^{MN}F^{OP}F^{QR}F_{MQ}F_{OR}F_{NP}-8F^{MN}F^{OP}F^{QR}F_{MQ}F_{NO}F_{PR}\Big],\end{aligned}\tag{4.36}$$

where $M,N,O,P,Q,R=1,\ldots 12$. Let us now decompose the directions $x^M=(x^i,x^\#)$ $(i,j,k.\cdots=1,2,\ldots,11$ and $\#=12)$ for all terms in (4.36).

The first term becomes

$$\begin{aligned}\mathrm{Tr}\left[F^{MN}F_{MN}\left(F^{OP}F^{QR}\right)^2\right]&=\mathrm{Tr}\Big[F^{ij}F_{ij}F^{kl}F^{mn}F_{kl}F_{mn}+2F^{ij}F_{ij}\left(F^{kl}F^{m\#}F_{kl}F_{m\#}+F^{k\#}F^{mn}F_{k\#}F_{mn}\right)+4F^{ij}F_{ij}F^{k\#}F^{m\#}F_{k\#}F_{m\#}\\
&\quad+2F^{i\#}F_{i\#}F^{kl}F^{mn}F_{kl}F_{mn}+4F^{i\#}F_{i\#}\left(F^{kl}F^{m\#}F_{kl}F_{m\#}+F^{k\#}F^{mn}F_{k\#}F_{mn}\right)+8F^{i\#}F_{i\#}F^{k\#}F^{m\#}F_{k\#}F_{m\#}\Big].\end{aligned}\tag{4.37a}$$

The second term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F_{MN}F^{OP}F^{QR}F_{QR}F_{OP}\right] \\
&\quad= \mathrm{Tr}\Big[F^{ij}F_{ij}F^{kl}F^{mn}F_{mn}F_{kl} + 2F^{ij}F_{ij}F^{k\#}F^{mn}F_{mn}F_{k\#} + 4F^{ij}F_{ij}F^{kl}F^{m\#}F_{m\#}F_{kl} \\
&\qquad + 4F^{i\#}F_{i\#}F^{kl}F^{m\#}F_{m\#}F_{kl} + 8F^{ij}F_{ij}F^{k\#}F^{m\#}F_{m\#}F_{k\#} + 8F^{i\#}F_{i\#}F^{k\#}F^{m\#}F_{m\#}F_{k\#}\Big].
\end{aligned} \tag{4.37b}$$

The third therm becomes

$$\mathrm{Tr}\left(F^{MN}F^{OP}F^{QR}\right)^2 = \mathrm{Tr}\left[F^{ij}F^{kl}F^{mn}F_{ij}F_{kl}F_{mn} + 6F^{ij}F^{kl}F^{m\#}F_{ij}F_{kl}F_{m\#} + 12F^{ij}F^{k\#}F^{m\#}F_{ij}F_{k\#}F_{m\#} + 8F^{i\#}F^{k\#}F^{m\#}F_{i\#}F_{k\#}F_{m\#}\right]. \tag{4.37c}$$

The fourth term becomes

$$\begin{aligned}
\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MN}F_{QR}F_{OP}\right] &= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{ij}F_{mn}F_{kl} + 2F^{i\#}F_{kl}F^{mn}F_{i\#}F_{mn}F_{kl} + 4F^{ij}F^{kl}F^{m\#}F_{ij}F_{m\#}F_{kl} \\
&\quad + 4F^{ij}F^{k\#}F^{m\#}F_{ij}F_{m\#}F_{k\#} + 8F^{i\#}F^{kl}F^{m\#}F_{i\#}F_{m\#}F_{kl} + 8F^{i\#}F^{k\#}F^{m\#}F_{i\#}F_{m\#}F_{k\#}\Big].
\end{aligned} \tag{4.37d}$$

The fifth term becomes

$$\begin{aligned}
\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{OP}F_{MN}F_{QR}\right] &= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{kl}F_{ij}F_{mn} + 2F^{ij}F^{kl}F^{m\#}F_{kl}F_{ij}F_{m\#} + 4F^{ij}F^{k\#}F^{mn}F_{k\#}F_{ij}F_{mn} \\
&\quad + 4F^{i\#}F^{k\#}F^{mn}F_{k\#}F_{i\#}F_{mn} + 8F^{ij}F^{k\#}F^{m\#}F_{k\#}F_{ij}F_{m\#} + 8F^{i\#}F^{k\#}F^{m\#}F_{k\#}F_{i\#}F_{m\#}\Big].
\end{aligned} \tag{4.37e}$$

The sixth term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F_{MN}F^{OP}F^{QR}F_{OQ}F_{PR}\right] \\
&\quad= \mathrm{Tr}\Big[F^{ij}F_{ij}F^{kl}F^{ml}F_{km}F_{ln} + F^{ij}F_{ij}F^{kl}F^{m\#}\left(F_{km}F_{l\#} - F_{k\#}F_{lm}\right) \\
&\qquad - F^{ij}F_{ij}F^{k\#}F^{mn}\left(F_{km}F_{n\#} - F_{m\#}F_{kn}\right) + F^{ij}F_{ij}F^{k\#}F^{m\#}\left(F_{k\#}F_{m\#} + F_{m\#}F_{k\#}\right) \\
&\qquad + 2F^{i\#}F_{i\#}F^{kl}F^{mn}F_{km}F_{ln} + 2F^{i\#}F_{i\#}F^{kl}F^{m\#}\left(F_{km}F_{l\#} - F_{k\#}F_{lm}\right) \\
&\qquad - 2F^{i\#}F_{i\#}F^{k\#}F^{mn}\left(F_{km}F_{n\#} - F_{m\#}F_{kn}\right) + 2F^{i\#}F_{i\#}F^{k\#}F^{m\#}\left(F_{k\#}F_{m\#} + F_{m\#}F_{k\#}\right)\Big].
\end{aligned} \tag{4.37f}$$

The seventh term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MN}F_{OQ}F_{PR}\right] \\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{ij}F_{km}F_{ln} + F^{ij}F^{kl}F^{m\#}F_{ij}\left(F_{km}F_{l\#} - F_{k\#}F_{lm}\right) \\
&\qquad - F^{ij}F^{k\#}F^{mn}F_{ij}\left(F_{km}F_{n\#} - F_{m\#}F_{kn}\right) + F^{ij}F^{k\#}F^{m\#}F_{ij}\left(F_{k\#}F_{m\#} + F_{m\#}F_{k\#}\right) \\
&\qquad + 2F^{i\#}F^{kl}F^{mn}F_{i\#}F_{km}F_{ln} + 2F^{i\#}F^{kl}F^{m\#}F_{i\#}\left(F_{km}F_{l\#} - F_{k\#}F_{lm}\right) \\
&\qquad - 2F^{i\#}F^{k\#}F^{mn}F_{i\#}\left(F_{km}F_{n\#} - F_{m\#}F_{kn}\right) + 2F^{i\#}F^{k\#}F^{m\#}F_{i\#}\left(F_{k\#}F_{m\#} + F_{m\#}F_{k\#}\right)\Big].
\end{aligned} \tag{4.37g}$$

The eighth term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{OP}F_{MQ}F_{NR}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{kl}F_{im}F_{jn} + F^{ij}F^{kl}F^{m\#}F_{kl}\left(F_{im}F_{j\#} - F_{i\#}F_{jm}\right)\\
&\qquad\qquad + 2F^{ij}F^{k\#}F^{mn}F_{k\#}F_{im}F_{jn} + 2F^{ij}F^{k\#}F^{m\#}F_{k\#}\left(F_{im}F_{j\#} - F_{i\#}F_{jm}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}F_{kl}\left(F_{im}F_{n\#} - F_{m\#}F_{in}\right) + F^{i\#}F^{kl}F^{m\#}F_{kl}\left(F_{i\#}F_{m\#} + F_{m\#}F_{i\#}\right)\\
&\qquad\qquad - 2F^{i\#}F^{k\#}F^{mn}F_{k\#}\left(F_{im}F_{n\#} - F_{m\#}F_{in}\right) + 2F^{i\#}F^{k\#}F^{m\#}F_{k\#}\left(F_{i\#}F_{m\#} + F_{m\#}F_{i\#}\right)\Big].
\end{aligned} \tag{4.37h}$$

The ninth term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MQ}F_{OP}F_{NR}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{im}F_{kl}F_{jn} + F^{ij}F^{kl}F^{m\#}\left(F_{im}F_{kl}F_{j\#} - F_{i\#}F_{kl}F_{jm}\right)\\
&\qquad\qquad + 2F^{ij}F^{k\#}F^{mn}F_{im}F_{k\#}F_{jn} + 2F^{ij}F^{k\#}F^{m\#}\left(F_{im}F_{k\#}F_{j\#} - F_{i\#}F_{k\#}F_{jm}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{im}F_{kl}F_{n\#} - F_{m\#}F_{kl}F_{in}\right) + F^{i\#}F^{kl}F^{m\#}\left(F_{i\#}F_{kl}F_{m\#} + F_{m\#}F_{kl}F_{i\#}\right)\\
&\qquad\qquad - 2F^{i\#}F^{k\#}F^{mn}\left(F_{im}F_{k\#}F_{n\#} - F_{m\#}F_{k\#}F_{in}\right) + 2F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{k\#}F_{m\#} + F_{m\#}F_{k\#}F_{i\#}\right)\Big]
\end{aligned} \tag{4.37i}$$

The tenth term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MO}F_{QR}F_{NP}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{ik}F_{mn}F_{jl} + 2F^{ij}F^{kl}F^{m\#}F_{ik}F_{m\#}F_{jl}\\
&\qquad\qquad + F^{ij}F^{k\#}F^{mn}\left(F_{ik}F_{mn}F_{j\#} - F_{i\#}F_{mn}F_{jk}\right) + 2F^{ij}F^{k\#}F^{m\#}\left(F_{ik}F_{m\#}F_{j\#} - F_{i\#}F_{m\#}F_{jk}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{ik}F_{mn}F_{l\#} - F_{k\#}F_{mn}F_{il}\right) - 2F^{i\#}F^{kl}F^{m\#}\left(F_{ik}F_{m\#}F_{l\#} - F_{k\#}F_{m\#}F_{il}\right)\\
&\qquad\qquad + F^{i\#}F^{k\#}F^{mn}\left(F_{i\#}F_{mn}F_{k\#} + F_{k\#}F_{mn}F_{i\#}\right) + 2F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{m\#}F_{k\#} + F_{k\#}F_{m\#}F_{i\#}\right)\Big].
\end{aligned} \tag{4.37j}$$

The eleventh term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MQ}F_{NR}F_{OP}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{im}F_{jn}F_{kl} + F^{ij}F^{kl}F^{m\#}\left(F_{im}F_{j\#} - F_{i\#}F_{jm}\right)F_{kl}\\
&\qquad\qquad + 2F^{ij}F^{k\#}F^{mn}F_{im}F_{jn}F_{k\#} + 2F^{ij}F^{k\#}F^{m\#}\left(F_{im}F_{j\#} - F_{i\#}F_{jm}\right)F_{k\#}\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{im}F_{n\#} - F_{m\#}F_{in}\right)F_{kl} + F^{i\#}F^{kl}F^{m\#}\left(F_{i\#}F_{m\#} + F_{m\#}F_{i\#}\right)F_{kl}\\
&\qquad\qquad - 2F^{i\#}F^{k\#}F^{mn}\left(F_{im}F_{n\#} - F_{m\#}F_{in}\right)F_{k\#} + 2F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{m\#} + F_{m\#}F_{i\#}\right)F_{k\#}\Big].
\end{aligned} \tag{4.37k}$$

The twelfth term becomes

$$\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MO}F_{NQ}F_{PR}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{ik}F_{jm}F_{ln} + F^{ij}F^{kl}F^{m\#}F_{ik}\left(F_{jm}F_{l\#} - F_{j\#}F_{lm}\right)\\
&\qquad\qquad - F^{ij}F^{k\#}F^{mn}\left(F_{ik}F_{jm}F_{n\#} + F_{i\#}F_{jm}F_{kn}\right) + F^{ij}F^{k\#}F^{m\#}\left(F_{ik}F_{j\#}F_{m\#} - F_{i\#}F_{jm}F_{k\#} + F_{i\#}F_{j\#}F_{km}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{ik}F_{m\#}F_{ln} - F_{k\#}F_{im}F_{ln}\right) - F^{i\#}F^{kl}F^{m\#}\left(F_{ik}F_{m\#}F_{l\#} - F_{k\#}F_{im}F_{l\#} + F_{k\#}F_{i\#}F_{lm}\right)\\
&\qquad + F^{i\#}F^{k\#}F^{mn}\left(F_{ik}F_{m\#}F_{n\#} + F_{i\#}F_{m\#}F_{kn} - F_{k\#}F_{im}F_{n\#}\right) + F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{m\#}F_{k\#} + F_{k\#}F_{i\#}F_{m\#}\right)\Big].
\end{aligned} \tag{4.37l}$$

The thirteenth term becomes

$$
\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MO}F_{PQ}F_{NR}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{ik}F_{lm}F_{jn} + F^{ij}F^{kl}F^{m\#}F_{ik}\left(F_{lm}F_{j\#} - F_{l\#}F_{jm}\right)\\
&\qquad - F^{ij}F^{k\#}F^{mn}\left(F_{ik}F_{m\#}F_{jn} + F_{i\#}F_{km}F_{jn}\right) - F^{ij}F^{k\#}F^{m\#}\left(F_{ik}F_{m\#}F_{j\#} + F_{i\#}F_{km}F_{j\#} - F_{i\#}F_{k\#}F_{jm}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{ik}F_{lm}F_{n\#} - F_{k\#}F_{lm}F_{in}\right) + F^{i\#}F^{kl}F^{m\#}\left(F_{ik}F_{l\#}F_{m\#} + F_{k\#}F_{lm}F_{i\#} - F_{k\#}F_{l\#}F_{im}\right)\\
&\qquad + F^{i\#}F^{k\#}F^{mn}\left(F_{ik}F_{m\#}F_{n\#} + F_{i\#}F_{km}F_{n\#} - F_{k\#}F_{m\#}F_{in}\right) - F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{k\#}F_{m\#} + F_{k\#}F_{m\#}F_{i\#}\right)\Big].
\end{aligned}
\tag{4.37m}
$$

The fourteenth term becomes

$$
\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{OQ}F_{MP}F_{NR}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{km}F_{il}F_{jn} + F^{ij}F^{kl}F^{m\#}\left(F_{km}F_{il}F_{j\#} - F_{k\#}F_{il}F_{jm}\right)\\
&\qquad + F^{ij}F^{k\#}F^{mn}\left(F_{km}F_{i\#} + F_{m\#}F_{ik}\right)F_{jn} + F^{ij}F^{k\#}F^{m\#}\left(F_{km}F_{i\#}F_{j\#} - F_{k\#}F_{i\#}F_{jm} + F_{m\#}F_{ik}F_{j\#}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}F_{km}\left(F_{il}F_{n\#} - F_{l\#}F_{in}\right) + F^{i\#}F^{kl}F^{m\#}\left(F_{k\#}F_{il}F_{m\#} + F_{km}F_{l\#}F_{i\#} - F_{k\#}F_{l\#}F_{im}\right)\\
&\qquad - F^{i\#}F^{k\#}F^{mn}\left(F_{km}F_{i\#}F_{n\#} + F_{m\#}F_{ik}F_{n\#} - F_{m\#}F_{k\#}F_{in}\right) + F^{i\#}F^{k\#}F^{m\#}\left(F_{k\#}F_{i\#}F_{m\#} + F_{m\#}F_{k\#}F_{i\#}\right)\Big].
\end{aligned}
\tag{4.37n}
$$

The fifteenth term becomes

$$
\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{OQ}F_{MR}F_{NP}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{km}F_{in}F_{jl} + F^{ij}F^{kl}F^{m\#}\left(F_{km}F_{i\#} - F_{k\#}F_{im}\right)F_{jl}\\
&\qquad + F^{ij}F^{k\#}F^{mn}\left(F_{km}F_{in}F_{j\#} + F_{m\#}F_{in}F_{jk}\right) + F^{ij}F^{k\#}F^{m\#}\left(F_{km}F_{i\#}F_{j\#} - F_{k\#}F_{im}F_{j\#} + F_{m\#}F_{i\#}F_{jk}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}F_{km}\left(F_{in}F_{l\#} - F_{n\#}F_{il}\right) - F^{i\#}F^{kl}F^{m\#}\left(F_{km}F_{i\#}F_{l\#} - F_{k\#}F_{im}F_{l\#} + F_{k\#}F_{m\#}F_{il}\right)\\
&\qquad - F^{i\#}F^{k\#}F^{mn}\left(F_{m\#}F_{in}F_{k\#} - F_{km}F_{n\#}F_{i\#} - F_{m\#}F_{n\#}F_{ik}\right) - F^{i\#}F^{k\#}F^{m\#}\left(F_{m\#}F_{i\#}F_{k\#} + F_{k\#}F_{m\#}F_{i\#}\right)\Big].
\end{aligned}
\tag{4.37o}
$$

The sixteenth term becomes

$$
\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MQ}F_{OR}F_{NP}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{im}F_{kn}F_{jl} + F^{ij}F^{kl}F^{m\#}\left(F_{im}F_{k\#} - F_{i\#}F_{km}\right)F_{jl}\\
&\qquad + F^{ij}F^{k\#}F^{mn}F_{im}\left(F_{kn}F_{j\#} + F_{n\#}F_{jk}\right) + F^{ij}F^{k\#}F^{m\#}\left(F_{im}F_{k\#}F_{j\#} - F_{i\#}F_{km}F_{j\#} - F_{i\#}F_{m\#}F_{jk}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{im}F_{kn}F_{l\#} - F_{m\#}F_{kn}F_{il}\right) - F^{i\#}F^{kl}F^{m\#}\left(F_{im}F_{k\#}F_{l\#} - F_{i\#}F_{km}F_{l\#} - F_{m\#}F_{k\#}F_{il}\right)\\
&\qquad - F^{i\#}F^{k\#}F^{mn}\left(F_{im}F_{n\#}F_{k\#} - F_{m\#}F_{kn}F_{i\#} - F_{m\#}F_{n\#}F_{ik}\right) + F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{m\#}F_{k\#} + F_{m\#}F_{k\#}F_{i\#}\right)\Big].
\end{aligned}
\tag{4.37p}
$$

The seventeenth term becomes

$$
\begin{aligned}
&\mathrm{Tr}\left[F^{MN}F^{OP}F^{QR}F_{MQ}F_{NO}F_{PR}\right]\\
&\quad= \mathrm{Tr}\Big[F^{ij}F^{kl}F^{mn}F_{im}F_{jk}F_{in} + F^{ij}F^{kl}F^{m\#}\left(F_{im}F_{jk}F_{l\#} - F_{i\#}F_{jk}F_{lm}\right)\\
&\qquad - F^{ij}F^{k\#}F^{mn}F_{im}\left(F_{jk}F_{n\#} + F_{j\#}F_{kn}\right) + F^{ij}F^{k\#}F^{m\#}\left(F_{i\#}F_{jk}F_{m\#} - F_{im}F_{j\#}F_{k\#} + F_{i\#}F_{j\#}F_{km}\right)\\
&\qquad - F^{i\#}F^{kl}F^{mn}\left(F_{im}F_{k\#} - F_{m\#}F_{ik}\right)F_{ln} - F^{i\#}F^{kl}F^{m\#}\left(F_{im}F_{k\#}F_{l\#} - F_{i\#}F_{k\#}F_{lm} - F_{m\#}F_{ik}F_{l\#}\right)\\
&\qquad + F^{i\#}F^{k\#}F^{mn}\left(F_{im}F_{k\#}F_{n\#} - F_{m\#}F_{ik}F_{n\#} - F_{m\#}F_{i\#}F_{kn}\right) - F^{i\#}F^{k\#}F^{m\#}\left(F_{i\#}F_{k\#}F_{m\#} + F_{m\#}F_{i\#}F_{k\#}\right)\Big]
\end{aligned}
\tag{4.37q}
$$

We calculate the summation for the whole terms, and then can check that the sixth-order terms (for instance, $F^{i\#}F_{i\#}F^{k\#}F^{m\#}F_{k\#}F_{m\#}$) and the eighth-order terms (for instance,$F^{i\#}F_{i\#}F^{kl}F^{m\#}F_{kl}F_{m\#}$) become zero. Here the "m-order term" means the term that contain m's roman indeces. Next we consider the tenth-order terms (for instance, $F^{ij}F_{ij}F^{kl}F^{m\#}F_{kl}F_{m\#}$), but this calculation is so long. Hence we give the conclusion of the calculation in the following lists.

Types	Coefficients	Using (3.27) (where $x^4 \to x^\#$) and integrate the $x^\#$-direction.
I	$4 = 4$	$F^{ij}F_{ij}F^{kl}F_{m\#}F^{kl}F_{m\#} = c_1[R_i, R_j]^2\left([R_k, R_l]R_m\right)^2,$
II	$4 = 4$	$F^{ij}F_{ij}F^{k\#}F^{mn}F_{k\#}F_{mn} = c_1[R_i, R_j]^2\left(R_k[R_m, R_n]\right)^2,$
III	$4 = 4$	$F^{i\#}F_{i\#}F^{kl}F^{mn}F_{kl}F_{mn} = c_1R_i^2\left([R_k, R_l][R_m, R_n]\right)^2,$
IV	$2 = 2$	$F^{ij}F_{ij}F^{k\#}F^{mn}F_{mn}F_{k\#} = c_1[R_i, R_j]^2R_k[R_m, R_n]^2R_k,$
V	$4 = 4$	$F^{ij}F_{ij}F^{kl}F^{m\#}F_{m\#}F_{kl} = c_1[R_i, R_j]^2[R_k, R_l]R_m^2[R_k, R_l],$
VI	$6 = 6$	$F^{ij}F^{kl}F^{m\#}F_{ij}F_{kl}F_{m\#} = c_1\left([R_i, R_j][R_k, R_l]R_m\right)^2,$
VII	$2 + 2 = 4$	$F^{i\#}F^{kl}F^{mn}F_{i\#}F_{mn}F_{kl} = c_1R_i[R_k, R_l][R_m, R_n]R_i[R_m, R_n][R_k, R_l],$
$VIII$	$4 + 4 = 8$	$F^{ij}F^{kl}F^{m\#}F_{ij}F_{m\#}F_{kl} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_j]R_m[R_k, R_l],$
IX	$-8 = -8$	$F^{ij}F_{ij}F^{kl}F^{m\#}F_{km}F_{l\#} = c_1[R_i, R_j]^2[R_k, R_l]R_m[R_k, R_m]R_l,$
X	$8 = 8$	$F^{ij}F_{ij}F^{kl}F^{m\#}F_{k\#}F_{lm} = c_1[R_i, R_j]^2[R_k, R_l]R_mR_k[R_l, R_m],$
XI	$8 = 8$	$F^{ij}F_{ij}F^{k\#}F^{mn}F_{km}F_{n\#} = c_1[R_i, R_j]^2R_k[R_m, R_n][R_k, R_m]R_n,$
XII	$-8 = -8$	$F^{ij}F_{ij}F^{k\#}F^{mn}F_{m\#}F_{kn} = c_1[R_i, R_j]^2R_k[R_m, R_n]R_m[R_k, R_n],$
$XIII$	$16 = 16$	$F^{i\#}F_{i\#}F^{kl}F^{mn}F_{km}F_{ln} = c_1R_i^2[R_k, R_l][R_m, R_n][R_k, R_m][R_l, R_n],$
XIV	$-8 = -8$	$F^{ij}F^{kl}F^{m\#}F_{ij}F_{km}F_{l\#} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_j][R_k.R_m]R_l,$
XV	$8 = 8$	$F^{ij}F^{kl}F^{m\#}F_{ij}F_{k\#}F_{lm} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_j]R_k[R_l, R_m],$
XVI	$8 = 8$	$F^{ij}F^{k\#}F^{mn}F_{ij}F_{km}F_{n\#} = c_1[R_i, R_j]R_k[R_m, R_n][R_i, R_j][R_k, R_m]R_n,$
$XVII$	$-8 = -8$	$F^{ij}F^{k\#}F^{mn}F_{ij}F_{m\#}F_{kn} = c_1[R_i, R_j]R_k[R_m, R_n][R_i, R_j]R_m[R_k, R_n],$
$XVIII$	$-16 = -16$	$F^{i\#}F^{kl}F^{mn}F_{i\#}F_{km}F_{ln} = c_1R_i[R_k, R_l][R_m, R_n]R_i[R_k, R_m][R_l, R_n],$
XIX	$-8 = -8$	$F^{ij}F^{kl}F^{m\#}F_{kl}F_{im}F_{j\#} = c_1[R_i, R_j][R_k, R_l]R_m[R_k, R_l][R_i, R_m]R_j,$
XX	$8 = 8$	$F^{ij}F^{kl}F^{m\#}F_{kl}F_{i\#}F_{jm} = c_1[R_i, R_j][R_k, R_l]R_m[R_k, R_l]R_i[R_j, R_m],$
XXI	$-16 = -16$	$F^{ij}F^{k\#}F^{mn}F_{k\#}F_{im}F_{jn} = c_1[R_i, R_j]R_k[R_m, R_n]R_k[R_i, R_m][R_j, R_n],$
$XXII$	$8 = 8$	$F^{i\#}F^{kl}F^{mn}F_{kl}F_{im}F_{n\#} = c_1R_i[R_k, R_l][R_m, R_n][R_k, R_l][R_i, R_m]R_n,$
$XXIII$	$-8 = -8$	$F^{i\#}F^{kl}F^{mn}F_{kl}F_{m\#}F_{in} = c_1R_i[R_k, R_l][R_m, R_n][R_k, R_l]R_m[R_i, R_n],$
$XXIV$	$-4 = -4$	$F^{ij}F^{kl}F^{m\#}F_{im}F_{kl}F_{j\#} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_m][R_k, R_l]R_j,$
XXV	$4 + 4 = 8$	$F^{ij}F^{kl}F^{m\#}F_{i\#}F_{kl}F_{jm} = c_1[R_i, R_j][R_k, R_l]R_mR_i[R_k, R_l][R_j, R_m],$
$XXVI$	$-8 = -8$	$F^{ij}F^{k\#}F^{mn}F_{im}F_{k\#}F_{jn} = c_1[R_i, R_j]R_k[R_m, R_n][R_i, R_m]R_k[R_j, R_n],$
$XXVII$	$-4 = -4$	$F^{i\#}F^{kl}F^{mn}F_{m\#}F_{kl}F_{in} = c_1R_i[R_k, R_l][R_m, R_n]R_m[R_k, R_l][R_i, R_n],$
$XXVIII$	$-8 - 8 = -16$	$F^{ij}F^{kl}F^{m\#}F_{ik}F_{m\#}F_{jl} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_k]R_m[R_j, R_l],$
$XXIX$	$-4 - 4 = -8$	$F^{ij}F^{k\#}F^{mn}F_{ik}F_{mn}F_{j\#} = c_1[R_i, R_j]R_k[R_m, R_n][R_i, R_k][R_m, R_n]R_j,$
XXX	$4 + 4 = 8$	$F^{ij}F^{k\#}F^{mn}F_{i\#}F_{mn}F_{jk} = c_1[R_i, R_j]R_k[R_m, R_n]R_i[R_m, R_n][R_j, R_k],$
$XXXI$	$4 + 4 = 8$	$F^{i\#}F^{kl}F^{mn}F_{ik}F_{mn}F_{l\#} = c_1R_i[R_k, R_l][R_m, R_n][R_i, R_k][R_m, R_n]R_l,$
$XXXII$	$-4 - 4 = -8$	$F^{i\#}F^{kl}F^{mn}F_{k\#}F_{mn}F_{il} = c_1R_i[R_k, R_l][R_m, R_n]R_k[R_m, R_n][R_i, R_l],$
$XXXIII$	$8 = 8$	$F^{ij}F^{kl}F^{m\#}F_{ik}F_{jm}F_{l\#} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_k][R_j, R_m]R_l,$
$XXXIV$	$-8 - 8 = -16$	$F^{ij}F^{kl}F^{m\#}F_{ik}F_{j\#}F_{lm} = c_1[R_i, R_j][R_k, R_l]R_m[R_i, R_k]R_j[R_l, R_m],$
$XXXV$	$-8 - 8 = -16$	$F^{ij}F^{k\#}F^{mn}F_{i\#}F_{jm}F_{kn} = c_1[R_i, R_j]R_k[R_m, R_n]R_i[R_j, R_m][R_k, R_n],$

Types	Coefficients	Using (3.27) (where $x^4 \to x^\#$) and integrate the $x^\#$-direction.
$XXXVI$	$8 = 8$	$F^{i\#}F^{kl}F^{mn}F_{k\#}F_{im}F_{ln} = c_1R_i[R_k,R_l][R_m,R_n]R_k[R_i,R_m][R_l,R_n]$,
$XXXVII$	$-8-8=-16$	$F^{ij}F^{kl}F^{m\#}F_{ik}F_{lm}F_{j\#} = c_1[R_i,R_j][R_k,R_l]R_m[R_i,R_k][R_l,R_m]R_j$,
$XXXVIII$	$8+8=16$	$F^{ij}F^{kl}F^{m\#}F_{ik}F_{l\#}F_{jm} = c_1[R_i,R_j][R_k,R_l]R_m[R_i,R_k]R_l[R_j,R_m]$,
$XXXIX$	$8+8=16$	$F^{ij}F^{k\#}F^{mn}F_{ik}F_{m\#}F_{jn} = c_1[R_i,R_j]R_k[R_m,R_n][R_i,R_k]R_m[R_j,R_n]$,
XL	$8+8=16$	$F^{ij}F^{k\#}F^{mn}F_{i\#}F_{km}F_{jn} = c_1[R_i,R_j]R_k[R_m,R_n]R_i[R_k,R_m][R_j,R_n]$,
XLI	$8+8=16$	$F^{i\#}F^{kl}F^{mn}F_{ik}F_{lm}F_{n\#} = c_1R_i[R_k,R_l][R_m,R_n][R_i,R_k][R_l,R_m]R_n$,
$XLII$	$-8-8=-16$	$F^{i\#}F^{kl}F^{mn}F_{k\#}F_{lm}F_{in} = c_1R_i[R_k,R_l][R_m,R_n]R_k[R_l,R_m][R_i,R_n]$,
$XLIII$	$8=8$	$F^{ij}F^{kl}F^{m\#}F_{km}F_{il}F_{j\#} = c_1[R_i,R_j][R_k,R_l]R_m[R_k,R_m][R_i,R_l]R_j$,
$XLIV$	$-8=-8$	$F^{ij}F^{kl}F^{m\#}F_{k\#}F_{il}F_{jm} = c_1[R_i,R_j][R_k,R_l]R_mR_k[R_i,R_l][R_j,R_m]$,
XLV	$8=8$	$F^{ij}F^{k\#}F^{mn}F_{km}F_{i\#}F_{jn} = c_1[R_i,R_j]R_k[R_m,R_n][R_k,R_m]R_i[R_j,R_n]$,
$XLVI$	$-8=-8$	$F^{ij}F^{k\#}F^{mn}F_{m\#}F_{in}F_{jk} = c_1[R_i,R_j]R_k[R_m,R_n]R_m[R_i,R_n][R_j,R_k]$,
$XLVII$	$-8-8=-16$	$F^{ij}F^{kl}F^{m\#}F_{km}F_{i\#}F_{jl} = c_1[R_i,R_j][R_k,R_l]R_m[R_k,R_m]R_i[R_j,R_l]$,
$XLVIII$	$8=8$	$F^{ij}F^{kl}F^{m\#}F_{k\#}F_{im}F_{jl} = c_1[R_i,R_j][R_k,R_l]R_mR_k[R_i,R_m][R_j,R_l]$,
$XLIX$	$-8-8=-16$	$F^{ij}F^{k\#}F^{mn}F_{m\#}F_{in}F_{jk} = c_1[R_i,R_j]R_k[R_m,R_n]R_m[R_i,R_n][R_j,R_k]$,
L	$8=8$	$F^{i\#}F^{kl}F^{mn}F_{km}F_{in}F_{l\#} = c_1R_i[R_k,R_l][R_m,R_n][R_k,R_m][R_i,R_n]R_l$,
LI	$8+8=16$	$F^{ij}F^{kl}F^{m\#}F_{im}F_{k\#}F_{jl} = c_1[R_i,R_j][R_k,R_l]R_m[R_i,R_m]R_k[R_j,R_l]$,
LII	$-8-8=-16$	$F^{ij}F^{kl}F^{m\#}F_{i\#}F_{km}F_{jl} = c_1[R_i,R_j][R_k,R_l]R_mR_i[R_k,R_m][R_j,R_l]$,
$LIII$	$8=8$	$F^{ij}F^{k\#}F^{mn}F_{im}F_{n\#}F_{jk} = c_1[R_i,R_j]R_k[R_m,R_n][R_i,R_m]R_n[R_j,R_k]$,
LIV	$8=8$	$F^{i\#}F^{kl}F^{mn}F_{m\#}F_{kn}F_{il} = c_1R_i[R_k,R_l][R_m,R_n]R_m[R_k,R_n][R_i,R_l]$.

Here we define c_1 as

$$c_1 := \int_{-\infty}^{\infty} dx^\# \, \frac{\psi_0^2(x^\#)}{2\sqrt{\pi}} \psi_+^4(x^\#) \left(\psi_+(x^\#) - 1\right)^4 . \tag{4.38}$$

The twelveth-order terms (for instance, $F^{ij}F_{ij}F^{kl}F^{mn}F_{kl}F_{mn}$) become

$$\begin{aligned}
&\int dx^\# \, \mathrm{Tr}\Big[2F^{ij}F_{ij}\left(F^{kl}F^{mn}\right)^2 + F^{ij}F_{ij}F^{kl}F^{mn}F_{mn}F_{kl} + \left(F^{ij}F^{kl}F^{mn}\right)^2 \\
&\qquad + F^{ij}F^{kl}F^{mn}F_{ij}F_{mn}F_{kl} + F^{ij}F^{kl}F^{mn}F_{kl}F_{ij}F_{mn} \\
&\quad - 8F^{ij}F_{ij}F^{kl}F^{mn}F_{km}F_{ln} - 8F^{ij}F^{kl}F^{mn}F_{ij}F_{km}F_{ln} - 8F^{ij}F^{kl}F^{mn}F_{kl}F_{im}F_{jn} \\
&\qquad - 4F^{ij}F^{kl}F^{mn}F_{im}F_{kl}F_{jn} - 4F^{ij}F^{kl}F^{mn}F_{ik}F_{mn}F_{jl} - 4F^{ij}F^{kl}F^{mn}F_{im}F_{jn}F_{kl} \\
&\quad + 8F^{ij}F^{kl}F^{mn}F_{ik}F_{jm}F_{ln} - 8F^{ij}F^{kl}F^{mn}F_{ik}F_{lm}F_{jn} + 8F^{ij}F^{kl}F^{mn}F_{km}F_{il}F_{jn} \\
&\qquad - 8F^{ij}F^{kl}F^{mn}F_{km}F_{in}F_{jl} + 8F^{ij}F^{kl}F^{mn}F_{im}F_{kn}F_{jl} - 8F^{ij}F^{kl}F^{mn}F_{im}F_{jk}F_{ln}\Big] \\
&= \mathrm{Tr}\Big[2c_2[R_i,R_j]^2\left([R_k,R_l][R_m,R_n]\right)^2 + c_2[R_i,R_j]^2[R_k,R_l][R_m,R_n]^2[R_k,R_l] + c_2\left([R_i,R_j][R_k,R_l][R_m,R_n]\right)^2 \\
&\qquad + c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_j][R_m,R_n][R_k,R_l] + c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_k,R_l][R_i,R_j][R_m,R_n] \\
&\quad - 8c_2[R_i,R_j]^2[R_k,R_l][R_m,R_n][R_k,R_m][R_l,R_n] - 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_j][R_k,R_m][R_l,R_n] \\
&\qquad - 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_k.R_l][R_i,R_m][R_j,R_n] - 4c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_k,R_l][R_j,R_n] \\
&\qquad\quad - 4c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_k][R_m,R_n][R_j,R_l] - 4c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_j,R_n][R_k,R_l] \\
&\quad + 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_k][R_j,R_m][R_l,R_n] - 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_k][R_l.R_m][R_j,R_n] \\
&\qquad + 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_k,R_m][R_i,R_l][R_j,R_n] - 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_k,R_m][R_i,R_n][R_j,R_l] \\
&\qquad\quad + 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_k,R_n][R_j,R_l] - 8c_2[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_j,R_k][R_i,R_n]\Big],
\end{aligned} \tag{4.39}$$

where

$$c_2 := \int_{-\infty}^{\infty} dx^\# \, \psi_+^6(x^\#) \left(\psi_+(x^\#) - 1\right)^6 . \tag{4.40}$$

Therefore we obtain the twelve-dimensional (static) Skyrme action as

$$E_{\mathrm{12dSky}} = \frac{1}{8}\int_{\mathbb{R}^{11}} d^{11}x \; c_2\mathcal{E}_{12} + c_1\mathcal{E}_{10}, \tag{4.41}$$

where $\mathcal{E}_{12}$ and $\mathcal{E}_{10}$ mean that the twelfth and tenth order term of R_i respectively.

$$
\begin{aligned}
\mathcal{E}_{12} \coloneqq \mathrm{Tr}\Big[&2[R_i,R_j]^2\,([R_k,R_l][R_m,R_n])^2 + [R_i,R_j]^2[R_k,R_l][R_m,R_n]^2[R_k,R_l] + \Big([R_i,R_j][R_k,R_l][R_m,R_n]\Big)^2 \\
&+ [R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_j][R_m,R_n][R_k,R_l] + [R_i,R_j][R_k,R_l][R_m,R_n][R_k,R_l][R_i,R_j][R_m,R_n] \\
&- 8[R_i,R_j]^2[R_k,R_l][R_m,R_n][R_k,R_m][R_l,R_n] - 8[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_j][R_k,R_m][R_l,R_n] \\
&- 8[R_i,R_j][R_k,R_l][R_m,R_n][R_k.R_l][R_i,R_m][R_j,R_n] - 4[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_k,R_l][R_j,R_n] \\
&- 4[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_k][R_m,R_n][R_j,R_l] - 4[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_j,R_n][R_k,R_l] \\
&+ 8[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_k][R_j,R_m][R_l,R_n] - 8[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_k][R_l.R_m][R_j,R_n] \\
&+ 8[R_i,R_j][R_k,R_l][R_m,R_n][R_k,R_m][R_i,R_l][R_j,R_n] - 8[R_i,R_j][R_k,R_l][R_m,R_n][R_k,R_m][R_i,R_n][R_j,R_l] \\
&+ 8[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_k,R_n][R_j,R_l] - 8[R_i,R_j][R_k,R_l][R_m,R_n][R_i,R_m][R_j,R_k][R_i,R_n]\Big],
\end{aligned}
\tag{4.42a}
$$

$$
\begin{aligned}
\mathcal{E}_{10} \coloneqq \mathrm{Tr}\Big[&4[R_i,R_j]^2\,([R_k,R_l]R_m)^2 + 4[R_i,R_j]^2\,(R_k[R_m,R_n])^2 + 4R_i^2\,([R_k,R_l][R_m,R_n])^2 + 2[R_i,R_j]^2R_k[R_m,R_n]^2R_k \\
&+ 4[R_i,R_j]^2[R_k,R_l]R_m^2[R_k,R_l] + 6\Big([R_i,R_j][R_k,R_l]R_m\Big)^2 + 4R_i[R_k,R_l][R_m,R_n]R_i[R_m,R_n][R_k,R_l] \\
&+ 8[R_i,R_j][R_k,R_l]R_m[R_i,R_j]R_m[R_k,R_l] \\
&- 8[R_i,R_j]^2[R_k,R_l]R_m[R_k,R_m]R_l + 8[R_i,R_j]^2[R_k,R_l]R_mR_k[R_l,R_m] + 8[R_i,R_j]^2R_k[R_m,R_n][R_k,R_m]R_n \\
&- 8[R_i,R_j]^2R_k[R_m,R_n]R_m[R_k,R_n] + 16R_i^2[R_k,R_l][R_m,R_n][R_k,R_m][R_l,R_n] - 8[R_i,R_j][R_k,R_l]R_m[R_i,R_j][R_k,R_m]R_l \\
&+ 8[R_i,R_j][R_k,R_l]R_m[R_i,R_j]R_k[R_l,R_m] + 8[R_i,R_j]R_k[R_m,R_n][R_i,R_j][R_k,R_m]R_n - 8[R_i,R_j]R_k[R_m,R_n][R_i,R_j]R_m[R_k,R_n] \\
&- 16R_i[R_k,R_l][R_m,R_n]R_i[R_k,R_m][R_l,R_n] - 8[R_i,R_j][R_k,R_l]R_m[R_k,R_l][R_i,R_m]R_j + 8[R_i,R_j][R_k,R_l]R_m[R_k,R_l]R_i[R_j,R_m] \\
&- 16[R_i,R_j]R_k[R_m,R_n]R_k[R_i,R_m][R_j,R_n] + 8R_i[R_k,R_l][R_m,R_n][R_k,R_l][R_i,R_m]R_n - 8R_i[R_k,R_l][R_m,R_n][R_k,R_l]R_m[R_i,R_n] \\
&- 4[R_i,R_j][R_k,R_l]R_m[R_i,R_m][R_k,R_l]R_j + 8[R_i,R_j][R_k,R_l]R_mR_i[R_k,R_l][R_j,R_m] - 8[R_i,R_j]R_k[R_m,R_n][R_i,R_m]R_k[R_j,R_n] \\
&- 4R_i[R_k,R_l][R_m,R_n]R_m[R_k,R_l][R_i,R_n] - 16[R_i,R_j][R_k,R_l]R_m[R_i,R_k]R_m[R_j,R_l] - 8[R_i,R_j]R_k[R_m,R_n][R_i,R_k][R_m,R_n]R_j \\
&+ 8[R_i,R_j]R_k[R_m,R_n]R_i[R_m,R_n][R_j,R_k] + 8R_i[R_k,R_l][R_m,R_n][R_i,R_k][R_m,R_n]R_l \\
&- 16[R_i,R_j][R_k,R_l]R_m[R_i,R_k]R_j[R_l,R_m] - 16[R_i,R_j]R_k[R_m,R_n]R_i[R_j,R_m][R_k,R_n] + 8R_i[R_k,R_l][R_m,R_n]R_k[R_i,R_m][R_l,R_n] \\
&- 16[R_i,R_j][R_k,R_l]R_m[R_i,R_k][R_l,R_m]R_j + 16[R_i,R_j][R_k,R_l]R_m[R_i,R_k]R_l[R_j,R_m] + 16[R_i,R_j]R_k[R_m,R_n][R_i,R_k]R_m[R_j,R_n] \\
&+ 16[R_i,R_j]R_k[R_m,R_n]R_i[R_k,R_m][R_j,R_n] + 16R_i[R_k,R_l][R_m,R_n][R_i,R_k][R_l,R_m]R_n - 16R_i[R_k,R_l][R_m,R_n]R_k[R_l,R_m][R_i,R_n] \\
&+ 8[R_i,R_j][R_k,R_l]R_m[R_k,R_m][R_i,R_l]R_j - 8[R_i,R_j][R_k,R_l]R_mR_k[R_i,R_l][R_j,R_m] + 8[R_i,R_j]R_k[R_m,R_n][R_k,R_m]R_i[R_j,R_n] \\
&- 8[R_i,R_j]R_k[R_m,R_n]R_m[R_i,R_n][R_j,R_k] - 16[R_i,R_j][R_k,R_l]R_m[R_k,R_m]R_i[R_j,R_l] + 8[R_i,R_j][R_k,R_l]R_mR_k[R_i,R_m][R_j,R_l] \\
&- 16[R_i,R_j]R_k[R_m,R_n]R_m[R_i,R_n][R_j,R_k] + 8R_i[R_k,R_l][R_m,R_n][R_k,R_m][R_i,R_n]R_l + 16[R_i,R_j][R_k,R_l]R_m[R_i,R_m]R_k[R_j,R_l] \\
&- 16[R_i,R_j][R_k,R_l]R_mR_i[R_k,R_m][R_j,R_l] + 8[R_i,R_j]R_k[R_m,R_n][R_i,R_m]R_n[R_j,R_k] + 8R_i[R_k,R_l][R_m,R_n]R_m[R_k,R_n][R_i,R_l]\Big].
\end{aligned}
\tag{4.42b}
$$

4.4 The detailed calculations

4.4.1 The Bogomol'nyi completion and the normalization constant in eight dimensions

We first lead the Bogomol'nyi completion in eight dimensions. Let us start at the following trivial equation which is analogy from four dimensional one (3.6).

$$\mathrm{Tr}\left(\left(\frac{\alpha}{3!}\right)^2 \varepsilon_{ijklmno}R_iR_jR_k \pm \beta R_{[l}R_mR_nR_{o]}\right)^2 \geq 0, \tag{4.43}$$

where α, β are real value constants which are defined as later. Expand the l.h.s. in the above equation:

$$\left(\left(\frac{\alpha}{3!}\right)^2 \varepsilon_{ijklmno}R_iR_jR_k \pm \beta R_{[l}R_mR_nR_{o]}\right)^2 = \left(\frac{\alpha}{3!}\right)^2 \varepsilon_{ijklmno}\varepsilon_{pqrlmno}R_iR_jR_kR_pR_qR_r \pm \frac{2}{3!}\alpha\beta\varepsilon_{ijklmno}R_iR_jR_kR_{[l}R_mR_nR_{o]} + \beta^2\left(R_{[l}R_mR_nR_{o]}\right)^2, \tag{4.44}$$

where $i, j, \ldots, p, r, r = 1, \ldots, 7$. The first term becomes

$$\left(\frac{1}{3!}\right)^2 \varepsilon_{ijklmno}\varepsilon_{pqrlmno}\mathrm{Tr}\left[R_iR_jR_kR_pR_qR_r\right] = \frac{1}{3!}\mathrm{Tr}\Big[\left([R_i,R_j]R_k\right)^2 - [R_i,R_j]R_k[R_i,R_k]R_j + [R_i,R_j]^2R_k^2 + [R_i,R_j]R_k[R_j,R_k]R_i + [R_j,R_k][R_i,R_j]R_kR_i - [R_i,R_k][R_i,R_j]R_kR_j\Big]. \tag{4.45}$$

Using the cyclic permutations of trace and the reparacement of indecies, we obtain

$$\mathrm{Tr}[R_i,R_j]R_k[R_j,R_k]R_i = -\mathrm{Tr}[R_i,R_j]R_i[R_k,R_j]R_k, \tag{4.46}$$

and

$$[R_j,R_k][R_i,R_j]R_kR_i - [R_i,R_k][R_i,R_j]R_kR_j = 2[R_i,R_j][R_k,R_i]R_jR_k. \tag{4.47}$$

Hence

$$\left(\frac{1}{3!}\right)^2 \varepsilon_{ijklmno}\varepsilon_{pqrlmno}\mathrm{Tr}\left[R_iR_jR_kR_pR_qR_r\right] = \frac{1}{3!}\mathrm{Tr}\Big[\left([R_i,R_j]R_k\right)^2 + [R_i,R_j]^2R_k^2 - [R_i,R_j]R_k[R_i,R_k]R_j + 2[R_i,R_j][R_k,R_i]R_jR_k - [R_i,R_j]R_i[R_k,R_j]R_k\Big]. \tag{4.48}$$

The third term becomes

$$\begin{aligned}\mathrm{Tr}\left(R_{[l}R_mR_nR_{o]}\right)^2 &= \frac{1}{4!}\left(\frac{1}{2!}\right)^2\mathrm{Tr}\Big[[R_i,R_j][R_k,R_l][R_i,R_j][R_k,R_l] + [R_i,R_j][R_k,R_l][R_k,R_l][R_i,R_j] - 4[R_i,R_j][R_k,R_l][R_i,R_k][R_j,R_l]\Big] \\ &= \frac{1}{4!}\frac{1}{4}\mathrm{Tr}\Big[\left([R_i,R_j][R_i,R_j]\right)^2 + \left([R_i,R_j][R_k,R_l]\right)^2 - 4[R_i,R_j][R_k,R_l][R_i,R_k][R_j,R_l]\Big].\end{aligned} \tag{4.49}$$

Without coefficients, (4.48) and (4.48) are just the 8th and 6th derivatives terms of the eight-dimensional Skyrme model respectively. Therefore we set the constants α, β as follows to accord with (4.5):

$$\alpha = 2\sqrt{3!c_1}, \quad \beta = 2\sqrt{4!c_2}, \tag{4.50}$$

and then the Bogomol'nyi bound which is the second term of (4.44) becomes

$$E_{\mathrm{Skyrme}} \geq \mp 16\sqrt{c_1c_2}\int d^7x\mathrm{Tr}\left[\varepsilon_{ijklmno}R_iR_jR_kR_lR_mR_nR_o\right], \tag{4.51}$$

where we using the relation: $\varepsilon_{ijklmno}R_iR_jR_kR_{[l}R_mR_nR_{o]} = \varepsilon_{ijklmno}R_iR_jR_kR_lR_mR_nR_o$. Here we set $c_1 = c_2 = 1$ then we obtain the Bogomol'nyi completion of the energy (4.9).

We next consider the normalization constant N_C. We determine the normalization constant by the condition that the topological charge of the single Skyrmion, becomes one. Using (4.63) and (4.4.2), we obtain

$$\begin{aligned}\varepsilon_{ijklmno}\mathrm{Tr}R_iR_jR_kR_lR_mR_nR_o &= \left(\frac{1}{2!}\right)^3 \varepsilon_{ijklmno}\mathrm{Tr}[R_i,R_j][R_k,R_l][R_m,R_n]R_o \\ &= \left(\frac{1}{2!}\right)^3 \varepsilon_{ijklmno}\mathrm{Tr}\Big[D^2\Sigma_{ij}\Sigma_{kl}\left(DB\Sigma_{mn}\hat{x}_o\hat{x}^\dagger + DC\Sigma_{mn}e_o^\dagger - EC\Theta_{mn}\hat{x}^\dagger e_o^\dagger - FA\Theta_{mn}e_o^\dagger\hat{x}^\dagger\right) \\ &\quad - D^2EC\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{mn}e_o^\dagger - D^2FA\Sigma_{ij}\Theta_{kl}\Sigma_{mn}e_o^\dagger\hat{x}^\dagger - ED^2C\Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{mn}e_o^\dagger - FD^2A\Theta_{ij}\Sigma_{kl}\Sigma_{mn}e_o^\dagger\hat{x}^\dagger\Big].\end{aligned} \tag{4.52}$$

The basis are

$$\begin{aligned}
&D^2\cdot DB \text{ term}: && \varepsilon_{ijklmno}\Sigma_{ij}\Sigma_{kl}\Sigma_{mn}\hat{x}_o\hat{x}^\dagger = && 5760\mathbf{1}_8,\\
&D^2\cdot DC \text{ term}: && \varepsilon_{ijklmno}\Sigma_{ij}\Sigma_{kl}\Sigma_{mn}e_o^\dagger = && 7\cdot 5760\mathbf{1}_8,\\
&D^2\cdot EC \text{ term}: && \varepsilon_{ijklmno}\Sigma_{ij}\Sigma_{kl}\Theta_{mn}\hat{x}^\dagger e_o^\dagger = && 5760\mathbf{1}_8,\\
&D^2\cdot FA \text{ term}: && \varepsilon_{ijklmno}\Sigma_{ij}\Sigma_{kl}\Theta_{mn}e_o^\dagger\hat{x}^\dagger = && -5760\mathbf{1}_8,\\
&D^2\cdot EC \text{ term}: && \varepsilon_{ijklmno}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{mn}e_o^\dagger = && 5760\mathbf{1}_8,\\
&D^2\cdot FA \text{ term}: && \varepsilon_{ijklmno}\Sigma_{ij}\Theta_{kl}\Sigma_{mn}e_o^\dagger\hat{x}^\dagger = && -5760\mathbf{1}_8,\\
&D^2\cdot EC \text{ term}: && \varepsilon_{ijklmno}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{mn}e_o^\dagger = && 5760\mathbf{1}_8,\\
&D^2\cdot FA \text{ term}: && \varepsilon_{ijklmno}\Theta_{ij}\Sigma_{kl}\Sigma_{mn}e_o^\dagger\hat{x}^\dagger = && -5760\mathbf{1}_8.
\end{aligned} \tag{4.53}$$

Hence

$$\varepsilon_{ijklmno}\mathrm{Tr}R_iR_jR_kR_lR_mR_nR_o = \left(\frac{1}{2!}\right)^3 5\cdot 5760 r^{-6}\sin^6 f\partial_r f\ \mathbf{1}_8. \tag{4.54}$$

Therefore we obtain

$$\begin{aligned}\int \varepsilon_{ijklmno}\mathrm{Tr}R_iR_jR_kR_lR_mR_nR_o\, d^7x &= \frac{16}{15}\pi^3\cdot\left(\frac{1}{2!}\right)^3\cdot 5760\mathrm{Tr}\mathbf{1}_8\cdot\int_0^\infty 5\sin^6 f\partial_r f\, dr \\ &= 30720\pi^3\times\frac{5!!}{6!!}\left(f(\infty)-f(0)\right) = -9600\pi^4.\end{aligned} \tag{4.55}$$

Here we have used the boundary condition $f(0)=\pi,\ f(\infty)=0$ and have taken into account the factor that comes from the six-dimensional spherical integration:

$$\int_{S^6} d\Omega_6 = \frac{16}{15}\pi^3 r^6, \tag{4.56}$$

where S^6 is the six-dimensional spherical surface and $d\Omega_6$ is the integral element of the six-dimensional sphere. Therefore the normalization constant is $N_C = -1/9600\pi^4$.

4.4.2 Hedgehog ansatz

We first introduce the hedgehog ansatz in $4n$ dimensions. Let $e_\mu^\dagger$ be the $(4n-1)$-dimensional ASD basis which is defined by (2.167), then we introduce the $4n$-dimensional hedgehog ansatz as

$$U(x^i) = \exp\left(f(r)\hat{x}^i e_i^\dagger\right), \tag{4.57}$$

where $\hat{x}^i := x^i/r,\ r^2 := x^ix^i$ and $f(r)$ is a real function, which is usually called as a profile function. The index i run from 1 to $4n-1$. For $e_i^\dagger = -e_i$, this hedgehog ansatz satisfies the relation $UU^\dagger = \mathbf{1}_{2^{2n-1}}$, thus $U\in \mathrm{U}(2^{2n-1})$. The right current R_i with the hedgehog ansatz becomes

$$U(x^i) = \cos f\mathbf{1}_{2^{2n-1}} + \sin f\hat{x}^\dagger, \tag{4.58a}$$

$$\partial_i U = -\sin f\partial_r f\hat{x}_i\mathbf{1}_{2^{2n-1}} + \left(\cos f\partial_r f - r^{-1}\sin f\right)\hat{x}_i\hat{x}^\dagger + r^{-1}\sin f e_i^\dagger, \tag{4.58b}$$

$$R_i = U\partial_i U^\dagger = r^{-1}\sin^2 f\hat{x}_i\mathbf{1}_{2^{2n-1}} - \left(-r^{-1}\sin f\cos f + \partial_r f\right)\hat{x}_i\hat{x}^\dagger - r^{-1}\sin f\cos f e_i^\dagger + r^{-1}\sin^2 f e_i^\dagger\hat{x}^\dagger, \tag{4.58c}$$

where $\hat{x}^\dagger := \hat{x}^i e_i^\dagger$ (and $\hat{x} := \hat{x}^i e_i$).

Using the lemma 2.4.1 and $e_i e_i^\dagger = e_i^\dagger e_i = (4n-1)\mathbf{1}_{2^{2n-1}}$, we obtain

$$e_i^\dagger \hat{x} e_i^\dagger = e_i^\dagger (2x_i \mathbf{1}_{2^{2n-1}} - e_i \hat{x}^\dagger) = 2\hat{x}^\dagger - (4n-1)\hat{x}^\dagger = -(4n-3)\hat{x}^\dagger, \tag{4.59a}$$

$$e_i^\dagger \hat{x}^\dagger e_i = e_i^\dagger (2x_i \mathbf{1}_{2^{2n-1}} - e_i^\dagger \hat{x}) = 2\hat{x}^\dagger + (4n-1)\hat{x} = -(4n-3)\hat{x}^\dagger, \tag{4.59b}$$

$$e_i^\dagger \hat{x}^\dagger e_i^\dagger = e_i^\dagger (-2x_i \mathbf{1}_{2^{2n-1}} - e_i^\dagger \hat{x}^\dagger) = -2\hat{x}^\dagger + (4n-1)\hat{x}^\dagger = (4n-3)\hat{x}^\dagger. \tag{4.59c}$$

Using these results, we give the useful following relations:

$$R_i R_i = -\left((\partial_r f)^2 + (4n-2) r^{-2} \sin^2 f\right) \mathbf{1}_{2^{2n-1}}, \tag{4.60a}$$

$$[R_i, R_j] = -r^{-2} \sin^2 f \Sigma_{ij} + 2\left(r^{-2} \sin^2 f - r^{-1} \sin f \cos f \partial_r f\right) \Theta_{ij} \hat{x}^\dagger - 2r^{-1} \sin^2 f \partial_r f \Theta_{ij}. \tag{4.60b}$$

where we define $\Sigma_{ij} := e_i^\dagger e_j - e_j^\dagger e_i = e_i e_j^\dagger - e_j e_i^\dagger$ (because of $e_i^\dagger = -e_i$) and $\Theta_{ij} := \hat{x}_i e_j^\dagger - \hat{x}_j e_i^\dagger$. Here the matrices Σ_{ij} and Θ_{ij} satisfy the following relations

$$\Theta_{ij} \hat{x}^\dagger = -\hat{x}^\dagger \Theta_{ij}, \qquad \Sigma_{ij} \Theta_{ij} = 8(2n-1)\hat{x}^\dagger, \qquad \Theta_{ij} \Sigma_{ij} = -8(2n-1)\hat{x}^\dagger,$$
$$\Theta_{ij}^2 = -4(2n-1)\, \mathbf{1}_{2^{2n-1}}, \qquad \left(\Sigma_{ij}\right)^2 = -4(4n-1)(4n-2)\, \mathbf{1}_{2^{2n-1}}. \tag{4.61}$$

The squares of the commutator $[R_i, R_j]$ is evaluated as

$$[R_i, R_j]^2 = -8(2n-1) r^{-2} \sin^2 f \left((4n-3) r^{-2} \sin^2 f + 2(\partial_r f)^2\right) \mathbf{1}_{2^{2n-1}}. \tag{4.62}$$

Derivation of the eight-dimensional Skyrme model with the hedgehog ansatz

Here we will lead the energy functional with the hedgehog ansatz (4.16). In the following, we use the seven-dimensional hedgehog ansatz thus the roman indeces i, j run from 1 to 7. For later convenience we rewrite the right current as

$$\begin{aligned} R_i &= r^{-1} \sin^2 f \hat{x}_i \mathbf{1}_8 - \left(-r^{-1} \sin f \cos f + \partial_r f\right) \hat{x}_i \hat{x}^\dagger - r^{-1} \sin f \cos f e_i^\dagger + r^{-1} \sin^2 f e_i^\dagger \hat{x}^\dagger \\ &=: A \hat{x}_i \mathbf{1}_8 - B \hat{x}_i \hat{x}^\dagger - C e_i^\dagger + A e_i^\dagger \hat{x}^\dagger, \end{aligned} \tag{4.63}$$

where $A := r^{-1} \sin^2 f$, $B := -r^{-1} \sin f \cos f + \partial_r f$ and $C := r^{-1} \sin f \cos f$. The square of right current is

$$R_i R_i = -\left((\partial_r f)^2 + 6 r^{-2} \sin^2 f\right) \mathbf{1}_8. \tag{4.64}$$

The commutator of the current R_i is

$$\begin{aligned} [R_i, R_j] &= -r^{-2} \sin^2 f \Sigma_{ij} + 2\left(r^{-2} \sin^2 f - r^{-1} \sin f \cos f \partial_r f\right) (\hat{x}_i e_j^\dagger - \hat{x}_j e_i^\dagger) \hat{x}^\dagger - 2r^{-1} \sin^2 f \partial_r f (\hat{x}_i e_j^\dagger - \hat{x}_j e_i^\dagger) \\ &=: -D \Sigma_{ij} + E \Theta_{ij} \hat{x}^\dagger - F \Theta_{ij}, \end{aligned} \tag{4.65}$$

where we have defined $D := r^{-2} \sin^2 f$, $E := 2\left(r^{-2} \sin^2 f - r^{-1} \sin f \cos f \partial_r f\right)$, $F := 2r^{-1} \sin^2 f \partial_r f$ and $\Theta_{ij} = \hat{x}_i e_j^\dagger - \hat{x}_j e_i^\dagger$. Now (4.61) becomes

$$\Theta_{ij} \hat{x}^\dagger = -\hat{x}^\dagger \Theta_{ij}, \qquad \Sigma_{ij} \Theta_{ij} = 24 \hat{x}^\dagger, \qquad \Theta_{ij} \Sigma_{ij} = -24 \hat{x}^\dagger,$$
$$\Theta_{ij}^2 = -12 \mathbf{1}_8, \qquad \left(\Sigma_{ij}\right)^2 = -168 \mathbf{1}_8. \tag{4.66}$$

The squares of the commutator $[R_i, R_j]$ is

$$[R_i, R_j]^2 = -24 r^{-2} \sin^2 f \left(5 r^{-2} \sin^2 f + 2(\partial_r f)^2\right) \mathbf{1}_8. \tag{4.67}$$

Using this result, we can calculate the first term in (4.5) as

$$\left([R_i, R_j]^2\right)^2 = 16 \cdot 6^2 r^{-4} \sin^4 f \left(25 r^{-4} \sin^4 f + 20 r^{-2} \sin^2 f (\partial_r f)^2 + 4(\partial_r f)^4\right) \mathbf{1}_8. \tag{4.68}$$

Things get more involved when we calculate the second term.

We expand the second term in (4.5) with the hedgehog ansatz as

$$
\begin{aligned}
\Big([R_i,R_j][R_k,R_l]\Big)^2 &= D^4\Sigma_{ij}\Sigma_{kl}\Sigma_{ij}\Sigma_{kl} - D^3E\Big(\Sigma_{ij}\Sigma_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Sigma_{ij}\Sigma_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl} + \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ij}\Sigma_{kl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ij}\Sigma_{kl}\Big)\\
&+ D^3F\Big(\Sigma_{ij}\Sigma_{kl}\Sigma_{ij}\Theta_{kl} + \Sigma_{ij}\Sigma_{kl}\Theta_{ij}\Sigma_{kl} + \Sigma_{ij}\Theta_{kl}\Sigma_{ij}\Sigma_{kl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ij}\Sigma_{kl}\Big)\\
&+ D^2(E^2+F^2)\Big(\Sigma_{ij}\Sigma_{kl}\Theta_{ij}\Theta_{kl} + \Theta_{ij}\Theta_{kl}\Sigma_{ij}\Sigma_{kl}\Big)\\
&+ D^2E^2\Big(\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ij}\hat{x}^\dagger\Sigma_{kl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Big)\\
&- D^2EF\Big(\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ij}\Theta_{kl} + \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ij}\Sigma_{kl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ij}\Theta_{kl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ij}\Sigma_{kl}\\
&\qquad + \Sigma_{ij}\Theta_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Sigma_{ij}\Theta_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Theta_{ij}\Sigma_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Big)\\
&- DE(E^2+F^2)\Big(\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ij}\Theta_{kl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ij}\Theta_{kl} + \Theta_{ij}\Theta_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Theta_{ij}\Theta_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Big)\\
&+ D^2F^2\Big(\Sigma_{ij}\Theta_{kl}\Sigma_{ij}\Theta_{kl} + \Sigma_{ij}\Theta_{kl}\Theta_{ij}\Sigma_{kl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ij}\Theta_{kl} + \Theta_{ij}\Sigma_{kl}\Theta_{ij}\Sigma_{kl}\Big)\\
&+ DF(E^2+F^2)\Big(\Sigma_{ij}\Theta_{kl}\Theta_{ij}\Theta_{kl} + \Theta_{ij}\Sigma_{kl}\Theta_{ij}\Theta_{kl} + \Theta_{ij}\Theta_{kl}\Sigma_{ij}\Theta_{kl} + \Theta_{ij}\Theta_{kl}\Theta_{ij}\Sigma_{kl}\Big) + (E^2+F^2)^2\Theta_{ij}\Theta_{kl}\Theta_{ij}\Theta_{kl}.
\end{aligned} \tag{4.69}
$$

Here we have used the relation $\Theta_{ij}\hat{x}^\dagger\Theta_{kl}\hat{x}^\dagger = \Theta_{ij}\Theta_{kl}$ and $\Theta_{ij}\hat{x}^\dagger\Theta_{kl} + \Theta_{ij}\Theta_{kl}\hat{x}^\dagger = 0$. We stress that terms that contain the odd number of $\hat{x}$ or $\hat{x}^\dagger$ vanish under the trace of the matrices. Since we need the trace of (4.69) in the energy functional, we neglect these terms and never calculate them in the following. Exploiting this fact, we are left with the terms that contain the even number of $\hat{x}$:

$$
\begin{aligned}
D^4 \text{ term}: &\quad \Sigma_{ij}\Sigma_{kl}\Sigma_{ij}\Sigma_{kl} = 1344\,\mathbf{1}_8,\\
D^3E \text{ term}: &\quad \Sigma_{ij}\Sigma_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \cdots + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ij}\Sigma_{kl} = 4\cdot 192\,\mathbf{1}_8,\\
D^2(E^2+F^2) \text{ term}: &\quad \Sigma_{ij}\Sigma_{kl}\Theta_{ij}\Theta_{kl} + \Theta_{ij}\Theta_{kl}\Sigma_{ij}\Sigma_{kl} = -2\cdot 384\,\mathbf{1}_8,\\
D^2E^2 \text{ term}: &\quad \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl} = 2\cdot 96\,\mathbf{1}_8,\\
&\quad \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ij}\hat{x}^\dagger\Sigma_{kl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger = -2\cdot 384\,\mathbf{1}_8,\\
DE(E^2+F^2) \text{ term}: &\quad \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ij}\Theta_{kl} + \cdots + \Theta_{ij}\Theta_{kl}\Theta_{ij}\hat{x}^\dagger\Sigma_{kl} = -4\cdot 192\,\mathbf{1}_8,\\
D^2F^2 \text{ term}: &\quad \Sigma_{ij}\Theta_{kl}\Sigma_{ij}\Theta_{kl} + \Theta_{ij}\Sigma_{kl}\Theta_{ij}\Sigma_{kl} = 2\cdot 864\,\mathbf{1}_8,\\
&\quad \Sigma_{ij}\Theta_{kl}\Theta_{ij}\Sigma_{kl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ij}\Theta_{kl} = -2\cdot 384\,\mathbf{1}_8,\\
(E^2+F^2)^2 \text{ term}: &\quad \Theta_{ij}\Theta_{kl}\Theta_{ij}\Theta_{kl} = -96\,\mathbf{1}_8.
\end{aligned} \tag{4.70}
$$

With this result at hand, we find that the second term in (4.5) becomes

$$
\mathrm{Tr}\Big([R_i,R_j][R_k,R_l]\Big)^2 = 1536r^{-4}\sin^4 f\Big(-5r^{-4}\sin^4 f + 20r^{-2}\sin^2 f(\partial_r f)^2 - 8(\partial_r f)^4\Big). \tag{4.71}
$$

We calculate the other terms by same method as follows.

Expand the third term in (4.5) with the hedgehog ansatz as

$$
\begin{aligned}
[R_i,R_j][R_k,R_l][R_i,R_k][R_j,R_l] &= D^4\Sigma_{ij}\Sigma_{kl}\Sigma_{ik}\Sigma_{jl} - D^3E\left(\Sigma_{ij}\Sigma_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Sigma_{ij}\Sigma_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl} + \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ik}\Sigma_{jl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ik}\Sigma_{jl}\right)\\
&+ D^3F\left(\Sigma_{ij}\Sigma_{kl}\Sigma_{ik}\Theta_{jl} + \Sigma_{ij}\Sigma_{kl}\Theta_{ik}\Sigma_{jl} + \Sigma_{ij}\Theta_{kl}\Sigma_{ik}\Sigma_{jl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ik}\Sigma_{jl}\right) + D^2(E^2+F^2)\left(\Sigma_{ij}\Sigma_{kl}\Theta_{ik}\Theta_{jl} + \Theta_{ij}\Theta_{kl}\Sigma_{ik}\Sigma_{jl}\right)\\
&+ D^2E^2\left(\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger\Sigma_{jl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl}\right)\\
&- D^2EF\Big(\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ik}\Theta_{jl} + \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ik}\Sigma_{jl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ik}\Theta_{jl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ik}\Sigma_{jl}\\
&\qquad + \Sigma_{ij}\Theta_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Sigma_{ij}\Theta_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Theta_{ij}\Sigma_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl}\Big)\\
&- DE(E^2+F^2)\left(\Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ik}\Theta_{jl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ik}\Theta_{jl} + \Theta_{ij}\Theta_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Theta_{ij}\Theta_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl}\right)\\
&+ D^2F^2\left(\Sigma_{ij}\Theta_{kl}\Sigma_{ik}\Theta_{jl} + \Sigma_{ij}\Theta_{kl}\Theta_{ik}\Sigma_{jl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ik}\Theta_{jl} + \Theta_{ij}\Sigma_{kl}\Theta_{ik}\Sigma_{jl}\right)\\
&+ DF(E^2+F^2)\left(\Sigma_{ij}\Theta_{kl}\Theta_{ik}\Theta_{jl} + \Theta_{ij}\Sigma_{kl}\Theta_{ik}\Theta_{jl} + \Theta_{ij}\Theta_{kl}\Sigma_{ik}\Theta_{jl} + \Theta_{ij}\Theta_{kl}\Theta_{ik}\Sigma_{jl}\right) + (E^2+F^2)^2\Theta_{ij}\Theta_{kl}\Theta_{ik}\Theta_{jl},
\end{aligned} \tag{4.72}
$$

and

$$
\begin{aligned}
D^4 \text{ term}: &\quad \Sigma_{ij}\Sigma_{kl}\Sigma_{ik}\Sigma_{jl} = && -12768\mathbf{1}_8,\\
D^3E \text{ term}: &\quad \Sigma_{ij}\Sigma_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \cdots + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ik}\Sigma_{jl} = && -4\cdot 1824\mathbf{1}_8,\\
D^2(E^2+F^2) \text{ term}: &\quad \Sigma_{ij}\Sigma_{kl}\Theta_{ik}\Theta_{jl} + \Theta_{ij}\Theta_{kl}\Sigma_{ik}\Sigma_{jl} = && 2\cdot 48\mathbf{1}_8,\\
D^2E^2 \text{ term}: &\quad \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl} = && -2\cdot 552\mathbf{1}_8\\
&\quad \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger\Sigma_{jl} + \Theta_{ij}\hat{x}^\dagger\Sigma_{kl}\Sigma_{ik}\Theta_{jl}\hat{x}^\dagger = && -2\cdot 312\mathbf{1}_8,\\
DE(E^2+F^2) \text{ term}: &\quad \Sigma_{ij}\Theta_{kl}\hat{x}^\dagger\Theta_{ik}\Theta_{jl} + \cdots + \Theta_{ij}\Theta_{kl}\Theta_{ik}\hat{x}^\dagger\Sigma_{jl} = && 4\cdot 24\mathbf{1}_8,\\
D^2F^2 \text{ term}: &\quad \Sigma_{ij}\Theta_{kl}\Sigma_{ik}\Theta_{jl} + \Theta_{ij}\Sigma_{kl}\Theta_{ik}\Sigma_{jl} = && -2\cdot 648\mathbf{1}_8,\\
&\quad \Sigma_{ij}\Theta_{kl}\Theta_{ik}\Sigma_{jl} + \Theta_{ij}\Sigma_{kl}\Sigma_{ik}\Theta_{jl} = && -2\cdot 312\mathbf{1}_8,\\
(E^2+F^2)^2 \text{ term}: &\quad \Theta_{ij}\Theta_{kj}\Theta_{ik}\Theta_{jl} = && 12\mathbf{1}_8.
\end{aligned}
\tag{4.73}
$$

Hence we obtain

$$
\mathrm{Tr}[R_i,R_j][R_k,R_l][R_i,R_k][R_j,R_l] = 768r^{-4}\sin^4 f\left(-55r^{-4}\sin^4 f - 80r^{-2}\sin^2 f(\partial_r f)^2 + 2(\partial_r f)^4\right). \tag{4.74}
$$

Using (4.64) and (4.67), we easily calculate the fourth term in (4.5) with the hedgehog ansatz:

$$
\mathrm{Tr}\left([R_i,R_j]\right)^2 R_k^2 = 192r^{-2}\sin^2 f\left(30r^{-4}\sin^4 f + 17r^{-2}\sin^2 f(\partial_r f)^2 + 2(\partial_r f)^4\right). \tag{4.75}
$$

Expand the fifth term in (4.5) with the hedgehog ansatz as

$$
\begin{aligned}
\left([R_i,R_j]R_k\right)^2 = &-D^2A^2\Sigma_{ij}\Sigma_{ij} - DA(FC+EA)\left(\Sigma_{ij}\Theta_{ij}\hat{x}^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{ij}\right)\\
&+\left(D^2B(B+2C)\right)\Sigma_{ij}\hat{x}^\dagger\Sigma_{ij}\hat{x}^\dagger + \left(DB(EB+EC)+DC(EB+FA)\right)\left(\Sigma_{ij}\hat{x}^\dagger\Theta_{ij} + \Theta_{ij}\Sigma_{ij}\hat{x}^\dagger\right)\\
&+D^2C^2\Sigma_{ij}e_k^\dagger\Sigma_{ij}e_k^\dagger - DC(EC-FA)\left(\Sigma_{ij}e_k^\dagger\Theta_{ij}\hat{x}^\dagger e_k^\dagger + \Theta_{ij}\hat{x}^\dagger e_k^\dagger\Sigma_{ij}e_k^\dagger\right)\\
&+A^2D^2\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Sigma_{ij}e_k^\dagger\hat{x}^\dagger - AD(EA+FC)\left(\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Theta_{ij}e_k^\dagger + \Theta_{ij}e_k^\dagger\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\right)\\
&+\left((E^2+F^2)(B^2-A^2+2BC)\right)\Theta_{ij}\Theta_{ij} + (EC-FA)^2\Theta_{ij}\hat{x}^\dagger e_k^\dagger\Theta_{ij}\hat{x}^\dagger e_k^\dagger + (EA+FC)^2\Theta_{ij}e_k^\dagger\Theta_{ij}e_k^\dagger + O(\hat{x}^\dagger),
\end{aligned}
\tag{4.76}
$$

where $O(\hat{x}^\dagger)$ means the term that contain the odd number of $\hat{x}$ or $\hat{x}^\dagger$.

$$
\begin{aligned}
D^2A^2 \text{ term}: &\quad \Sigma_{ij}\Sigma_{ij} = && -168\mathbf{1}_8,\\
DA(FC+EA) \text{ term}: &\quad \Sigma_{ij}\hat{x}^\dagger\Theta_{ij} + \Theta_{ij}\Sigma_{ij}\hat{x}^\dagger = && -2\cdot 24\mathbf{1}_8,\\
D^2B(B+2C) \text{ term}: &\quad \Sigma_{ij}\hat{x}^\dagger\Sigma_{ij}\hat{x}^\dagger = && 72\mathbf{1}_8,\\
DB(EB+EC)+DC(EB+FA) \text{ term}: &\quad \Sigma_{ij}\hat{x}^\dagger\Theta_{ij} + \Theta_{ij}\Sigma_{ij}\hat{x}^\dagger = && 2\cdot 24\mathbf{1}_8,\\
D^2C^2 \text{ term}: &\quad \Sigma_{ij}e_k^\dagger\Sigma_{ij}e_k^\dagger = && 504\mathbf{1}_8,\\
DC(EC-FA) \text{ term}: &\quad \Sigma_{ij}e_k^\dagger\Theta_{ij}\hat{x}^\dagger e_k^\dagger + \Theta_{ij}\hat{x}^\dagger e_k^\dagger\Sigma_{ij}e_k^\dagger = && 2\cdot 72\mathbf{1}_8,\\
A^2D^2 \text{ term}: &\quad \Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Sigma_{ij}e_k^\dagger\hat{x}^\dagger = && -120\mathbf{1}_8,\\
AD(EA+FC) \text{ term}: &\quad \Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Theta_{ij}e_k^\dagger + \Theta_{ij}e_k^\dagger\Sigma_{ij}e_k^\dagger\hat{x}^\dagger = && -2\cdot 72\mathbf{1}_8,\\
(E^2+F^2)(B^2-A^2+2BC) \text{ term}: &\quad \Theta_{ij}\Theta_{ij} = && -12\mathbf{1}_8,\\
(EC-FA)^2 \text{ term}: &\quad \Theta_{ij}\hat{x}^\dagger e_k^\dagger\Theta_{ij}\hat{x}^\dagger e_k^\dagger = && 36\mathbf{1}_8,\\
(EA+FC)^2 \text{ term}: &\quad \Theta_{ij}e_k^\dagger\Theta_{ij}e_k^\dagger = && -60\mathbf{1}_8.
\end{aligned}
\tag{4.77}
$$

Hence

$$
\mathrm{Tr}\left([R_i,R_j]R_k\right)^2 = 192r^{-2}\sin^2 f\left(10r^{-4}\sin^4 f + 13r^{-2}\sin^2 f(\partial_r f)^2 - 2(\partial_r f)^4\right). \tag{4.78}
$$

Expand the sixth term in (4.5) with the hedgehog ansatz as

$$\begin{aligned}
[R_i,R_j]R_k[R_i,R_k]R_j &= D^2A^2\hat{x}_j\hat{x}_k\Sigma_{ij}\Sigma_{ik} + D^2A^2\left(\hat{x}_k\Sigma_{ij}\Sigma_{ik}e_j^\dagger\hat{x}^\dagger + \hat{x}_j\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Sigma_{ik}\right) + DA(EA-FB)\left(\hat{x}_k\hat{x}_j\Sigma_{ij}\Theta_{ik}\hat{x}^\dagger + \hat{x}_k\hat{x}_j\Theta_{ij}\hat{x}^\dagger\Sigma_{ik}\right)\\
&- DA(EA+FC)\left(\hat{x}_k\Sigma_{ij}\Theta_{ik}e_j^\dagger + \hat{x}_j\Theta_{ij}e_k^\dagger\Sigma_{ik}\right) + D^2B^2\hat{x}_k\hat{x}_j\Sigma_{ij}\hat{x}^\dagger\Sigma_{ik}\hat{x}^\dagger + D^2BC\left(\hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Sigma_{ik}e_j^\dagger + \hat{x}_j\Sigma_{ij}e_k^\dagger\Sigma_{ik}\hat{x}^\dagger\right)\\
&+ DB(EB+FA)\left(\hat{x}_k\hat{x}_j\Sigma_{ij}\hat{x}^\dagger\Theta_{ik} + \hat{x}_j\hat{x}_k\Theta_{ij}\Sigma_{ik}\hat{x}^\dagger\right) - DB(EC-FA)\left(\hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger e_j^\dagger + \hat{x}_j\Theta_{ij}\hat{x}^\dagger e_k^\dagger\Sigma_{ik}\hat{x}^\dagger\right)\\
&+ D^2C^2\Sigma_{ij}e_k^\dagger\Sigma_{ik}e_j^\dagger + DC(EB+FA)\left(\hat{x}_j\Sigma_{ij}e_k^\dagger\Theta_{ik} + \hat{x}_k\Theta_{ij}\Sigma_{ik}e_j^\dagger\right) - DC(EC-FA)\left(\Sigma_{ij}e_k^\dagger\Theta_{ik}\hat{x}^\dagger e_j^\dagger + \Theta_{ij}\hat{x}^\dagger e_k^\dagger\Sigma_{ik}e_j^\dagger\right)\\
&+ A^2D^2\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Sigma_{ik}e_j^\dagger\hat{x}^\dagger + AD(EA-FB)\left(\hat{x}_j\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger + \hat{x}_k\Theta_{ij}\hat{x}^\dagger\Sigma_{ik}e_j^\dagger\hat{x}^\dagger\right)\\
&- AD(EA+FC)\left(\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Theta_{ik}e_j^\dagger + \Theta_{ij}e_k^\dagger\Sigma_{ik}e_j^\dagger\hat{x}^\dagger\right) + (EA-FB)^2\hat{x}_j\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger\\
&- (EA-FB)(EA+FC)\left(\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Theta_{ik}e_j^\dagger + \hat{x}_j\Theta_{ij}e_k^\dagger\Theta_{ik}\hat{x}^\dagger\right) + (EB+FA)^2\hat{x}_j\hat{x}_k\Theta_{ij}\Theta_{ik}\\
&- (EB+FA)(EC-FA)\left(\hat{x}_k\Theta_{ij}\Theta_{ik}\hat{x}^\dagger e_j^\dagger + \hat{x}_j\Theta_{ij}\hat{x}^\dagger e_k^\dagger\Theta_{ik}\right)\\
&+ (EC-FA)^2\Theta_{ij}\hat{x}^\dagger e_k^\dagger\Theta_{ik}\hat{x}^\dagger e_j^\dagger + (EA+FC)^2\Theta_{ij}e_k^\dagger\Theta_{ik}e_j^\dagger + O(\hat{x}^\dagger),
\end{aligned} \tag{4.79}$$

and

$$\begin{aligned}
&D^2A^2 \text{ term.}: & \hat{x}_j\hat{x}_k\Sigma_{ij}\Sigma_{ik} &= -24\mathbf{1}_8,\\
&D^2A^2 \text{ term.}: & \hat{x}_k\Sigma_{ij}\Sigma_{ik}e_j^\dagger\hat{x}^\dagger + \hat{x}_j\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Sigma_{ik} &= -2\cdot 96\mathbf{1}_8,\\
&DA(EA-FB) \text{ term.}: & \hat{x}_k\hat{x}_j\Sigma_{ij}\Theta_{ik}\hat{x}^\dagger + \hat{x}_k\hat{x}_j\Theta_{ij}\hat{x}^\dagger\Sigma_{ik} &= -2\cdot 12\mathbf{1}_8,\\
&DA(EA+FC) \text{ term.}: & \hat{x}_k\Sigma_{ij}\Theta_{ik}e_j^\dagger + \hat{x}_j\Theta_{ij}e_k^\dagger\Sigma_{ik} &= -2\cdot 72\mathbf{1}_8,\\
&D^2B^2 \text{ term.}: & \hat{x}_k\hat{x}_j\Sigma_{ij}\hat{x}^\dagger\Sigma_{ik}\hat{x}^\dagger &= -24\mathbf{1}_8,\\
&D^2BC \text{ term.}: & \hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Sigma_{ik}e_j^\dagger + \hat{x}_j\Sigma_{ij}e_k^\dagger\Sigma_{ik}\hat{x}^\dagger &= -2\cdot 144\mathbf{1}_8,\\
&DB(EB+FA) \text{ term.} & \hat{x}_k\hat{x}_j\Sigma_{ij}\hat{x}^\dagger\Theta_{ik} + \hat{x}_j\hat{x}_k\Theta_{ij}\Sigma_{ik}\hat{x}^\dagger &= 2\cdot 12\mathbf{1}_8,\\
&DB(EC-FA) \text{ term.}: & \hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger e_j^\dagger + \hat{x}_j\Theta_{ij}\hat{x}^\dagger e_k^\dagger\Sigma_{ik}\hat{x}^\dagger &= -2\cdot 72\mathbf{1}_8,\\
&D^2C^2 \text{ term.}: & \Sigma_{ij}e_k^\dagger\Sigma_{ik}e_j^\dagger &= -1008\mathbf{1}_8,\\
&DC(EB+FA) \text{ term.}: & \hat{x}_j\Sigma_{ij}e_k^\dagger\Theta_{ik} + \hat{x}_k\Theta_{ij}\Sigma_{ik}e_j^\dagger &= 2\cdot 12\mathbf{1}_8,\\
&DC(EC-FA) \text{ term.}: & \Sigma_{ij}e_k^\dagger\Theta_{ik}\hat{x}^\dagger e_j^\dagger + \Theta_{ij}\hat{x}^\dagger e_k^\dagger\Sigma_{ik}e_j^\dagger &= -2\cdot 144\mathbf{1}_8,\\
&A^2D^2 \text{ term.}: & \Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Sigma_{ik}e_j^\dagger\hat{x}^\dagger &= -240\mathbf{1}_8,\\
&AD(EA-FB) \text{ term.}: & \hat{x}_j\Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger + \hat{x}_k\Theta_{ij}\hat{x}^\dagger\Sigma_{ik}e_j^\dagger\hat{x}^\dagger &= 2\cdot 12\mathbf{1}_8,\\
&AD(EA+FC) \text{ term.}: & \Sigma_{ij}e_k^\dagger\hat{x}^\dagger\Theta_{ik}e_j^\dagger + \Theta_{ij}e_k^\dagger\Sigma_{ik}e_j^\dagger\hat{x}^\dagger &= 2\cdot 144\mathbf{1}_8,\\
&(EA-FB)^2 \text{ term.}: & \hat{x}_j\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Theta_{ik}\hat{x}^\dagger &= -6\mathbf{1}_8,\\
&(EA-FB)(EA+FC) \text{ term.}: & \hat{x}_k\Theta_{ij}\hat{x}^\dagger\Theta_{ik}e_j^\dagger + \hat{x}_j\Theta_{ij}e_k^\dagger\Theta_{ik}\hat{x}^\dagger &= -2\cdot 6\mathbf{1}_8,\\
&(EB+FA)^2 \text{ term.}: & \hat{x}_j\hat{x}_k\Theta_{ij}\Theta_{ik} &= -6\mathbf{1}_8,\\
&(EB+FA)(EC-FA) \text{ term.}: & \hat{x}_k\Theta_{ij}\Theta_{ik}\hat{x}^\dagger e_j^\dagger + \hat{x}_j\Theta_{ij}\hat{x}^\dagger e_k^\dagger\Theta_{ik} &= 2\cdot 6\mathbf{1}_8,\\
&(EC-FA)^2 \text{ term.}: & \Theta_{ij}\hat{x}^\dagger e_k^\dagger\Theta_{ik}\hat{x}^\dagger e_j^\dagger &= -42\mathbf{1}_8,\\
&(EA+FC)^2 \text{ term.}: & \Theta_{ij}e_k^\dagger\Theta_{ik}e_j^\dagger &= 30\mathbf{1}_8.
\end{aligned} \tag{4.80}$$

Hence

$$\mathrm{Tr}[R_i,R_j]R_k[R_i,R_k]R_j = 192r^{-2}\sin^2 f\left(-25r^{-4}\sin^4 f - 16r^{-2}\sin^2(\partial_r f)^2 - (\partial_r f)^4\right). \tag{4.81}$$

Expand the seventh term in (4.5) with the hedgehog ansatz as

$$\begin{aligned}[R_i, R_j][R_k, R_i]R_jR_k = {} & D^2(A^2 - B^2 - 2BC)\hat{x}_j\hat{x}_k\Sigma_{ij}\Sigma_{ki} + D^2(A^2 - BC)\Sigma_{ij}\Sigma_{ki}\Theta_{jk}\hat{x}^\dagger + D^2(C^2 + A^2)\Sigma_{ij}\Sigma_{ki}e_j^\dagger e_k^\dagger \\ & - DE(A^2 - B^2 - 2BC)\left(\hat{x}_j\hat{x}_k\Sigma_{ij}\Theta_{ki}\hat{x}^\dagger + \hat{x}_j\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Sigma_{ki}\right) - DE(A^2 - BC)\left(\Sigma_{ij}\Theta_{ki}\Theta_{jk} + \Theta_{ij}\hat{x}^\dagger\Sigma_{ki}\Theta_{jk}\hat{x}^\dagger\right) \\ & - DE(C^2 + A^2)\left(\Sigma_{ij}\Theta_{ki}\hat{x}^\dagger e_j^\dagger e_k^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{ki}e_j^\dagger e_k^\dagger\right) - DF(AC + BA)\left(\Sigma_{ij}\Theta_{ki}\Theta_{jk} + \Theta_{ij}\Sigma_{ki}\Theta_{jk}\right) \\ & + (E^2 + F^2)(A^2 - B^2 - 2BC)\hat{x}_j\hat{x}_k\Theta_{ij}\Theta_{ki} + (E^2 + F^2)(A^2 - BC)\Theta_{ij}\Theta_{ki}\Theta_{jk}\hat{x}^\dagger \\ & + (E^2 + F^2)(C^2 + A^2)\Theta_{ij}\Theta_{ki}e_j^\dagger e_k^\dagger + \mathcal{O}(\hat{x}^\dagger),\end{aligned} \tag{4.82}$$

and

$$\begin{aligned} D^2(A^2 - B^2 - 2BC)\text{ term.}: \quad & \hat{x}_j\hat{x}_k\Sigma_{ij}\Sigma_{ki} = & 24\mathbf{1}_8, \\ D^2(A^2 - BC)\text{ term.}: \quad & \Sigma_{ij}\Sigma_{ki}\Theta_{jk}\hat{x}^\dagger = & -240\mathbf{1}_8, \\ D^2(C^2 + A^2)\text{ term.}: \quad & \Sigma_{ij}\Sigma_{ki}e_j^\dagger e_k^\dagger = & 672\mathbf{1}_8, \\ DE(A^2 - B^2 - 2BC)\text{ term.}: \quad & \hat{x}_j\hat{x}_k\Sigma_{ij}\Theta_{ki}\hat{x}^\dagger + \hat{x}_j\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Sigma_{ki} = & 2\cdot 12\mathbf{1}_8 \\ DE(A^2 - BC)\text{ term.}: \quad & \Sigma_{ij}\Theta_{ki}\Theta_{jk} + \Theta_{ij}\hat{x}^\dagger\Sigma_{ki}\Theta_{jk}\hat{x}^\dagger = & -2\cdot 60\mathbf{1}_8, \\ DE(C^2 + A^2)\text{ term.}: \quad & \Sigma_{ij}\Theta_{ki}\hat{x}^\dagger e_j^\dagger e_k^\dagger + \Theta_{ij}\hat{x}^\dagger\Sigma_{ki}e_j^\dagger e_k^\dagger = & 2\cdot 96\mathbf{1}_8, \\ DF(AC + BA)\text{ term.}: \quad & \Sigma_{ij}\Theta_{ki}\Theta_{jk} + \Theta_{ij}\Sigma_{ki}\Theta_{jk} = & -2\cdot 60\mathbf{1}_8, \\ (E^2 + F^2)(A^2 - B^2 - 2BC): \quad & \hat{x}_j\hat{x}_k\Theta_{ij}\Theta_{ki} = & 6\mathbf{1}_8, \\ (E^2 + F^2)(A^2 - BC): \quad & \Theta_{ij}\Theta_{ki}\Theta_{jk}\hat{x}^\dagger = & 0, \\ (E^2 + F^2)(C^2 + A^2): \quad & \Theta_{ij}\Theta_{ki}e_j^\dagger e_k^\dagger = & 18\mathbf{1}_8. \end{aligned} \tag{4.83}$$

Hence

$$\mathrm{Tr}[R_i, R_j][R_k, R_i]R_jR_k = 192r^{-2}\sin^2 f\left(15r^{-4}\sin^4 f + 14r^{-2}\sin^2 f(\partial_r f)^2 - (\partial_r f)^4\right). \tag{4.84}$$

Expand the eighth term in (4.5) with the hedgehog ansatz as

$$\begin{aligned}[R_i, R_j]R_i[R_k, R_j]R_k = {} & D^2A^2\hat{x}_i\hat{x}_k\Sigma_{ij}\Sigma_{kj} + D^2A^2\left(\hat{x}_i\Sigma_{ij}\Sigma_{kj}e_k^\dagger\hat{x}^\dagger + \hat{x}_k\Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Sigma_{kj}\right) + DA(EA - FB)\left(\hat{x}_i\hat{x}_k\Sigma_{ij}\Theta_{kj}\hat{x}^\dagger + \hat{x}_i\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Sigma_{kj}\right) \\ & - DA(EA + FC)\left(\hat{x}_i\Sigma_{ij}\Theta_{kj}e_k^\dagger + \hat{x}_k\Theta_{ij}e_i^\dagger\Sigma_{kj}\right) + D^2B^2\hat{x}_i\hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Sigma_{kj}\hat{x}^\dagger + D^2BC\left(\hat{x}_i\Sigma_{ij}\hat{x}^\dagger\Sigma_{kj}e_k^\dagger + \hat{x}_k\Sigma_{ij}e_i^\dagger\Sigma_{kj}\hat{x}^\dagger\right) \\ & + DB(EB + FA)\left(\hat{x}_i\hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Theta_{kj} + \hat{x}_i\hat{x}_k\Theta_{ij}\Sigma_{kj}\hat{x}^\dagger\right) - DB(EC - FA)\left(\hat{x}_i\Sigma_{ij}\hat{x}^\dagger\Theta_{kj}\hat{x}^\dagger e_k^\dagger + \hat{x}_k\Theta_{ij}\hat{x}^\dagger e_i^\dagger\Sigma_{kj}\hat{x}^\dagger\right) \\ & + D^2C^2\Sigma_{ij}e_i^\dagger\Sigma_{kj}e_k^\dagger + DC(EB + FA)\left(\hat{x}_k\Sigma_{ij}e_i^\dagger\Theta_{kj} + \hat{x}_i\Theta_{ij}\Sigma_{kj}e_k^\dagger\right) - DC(EC - FA)\left(\Sigma_{ij}e_i^\dagger\Theta_{kj}\hat{x}^\dagger e_k^\dagger + \Theta_{ij}\hat{x}^\dagger e_i^\dagger\Sigma_{kj}e_k^\dagger\right) \\ & + A^2D^2\Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Sigma_{kj}e_k^\dagger\hat{x}^\dagger + AD(EA - FB)\left(\hat{x}_k\Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Theta_{kj}\hat{x}^\dagger + \hat{x}_i\Theta_{ij}\hat{x}^\dagger\Sigma_{kj}e_k^\dagger\hat{x}^\dagger\right) \\ & - AD(EA + FC)\left(\Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Theta_{kj}e_k^\dagger + \Theta_{ij}e_i^\dagger\Sigma_{kj}e_k^\dagger\hat{x}^\dagger\right) + (EA - FB)^2\hat{x}_i\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Theta_{kj}\hat{x}^\dagger \\ & - (EA - FB)(EA + FC)\left(\hat{x}_i\Theta_{ij}\hat{x}^\dagger\Theta_{kj}e_k^\dagger + \hat{x}_k\Theta_{ij}e_i^\dagger\Theta_{kj}\hat{x}^\dagger\right) + (EB + FA)^2\hat{x}_i\hat{x}_k\Theta_{ij}\Theta_{kj} \\ & - (EB + FA)(EC - FA)\left(\hat{x}_i\Theta_{ij}\Theta_{kj}\hat{x}^\dagger e_k^\dagger + \hat{x}_k\Theta_{ij}\hat{x}^\dagger e_i^\dagger\Theta_{kj}\right) \\ & + (EC - FA)^2\Theta_{ij}\hat{x}^\dagger e_i^\dagger\Theta_{kj}\hat{x}^\dagger e_k^\dagger + (EA + FC)^2\Theta_{ij}e_i^\dagger\Theta_{kj}e_k^\dagger + \mathcal{O}(\hat{x}^\dagger),\end{aligned} \tag{4.85}$$

and

$$
\begin{aligned}
D^2A^2 \text{ term}: &\quad \hat{x}_i\hat{x}_k\Sigma_{ij}\Sigma_{kj} = & -24\mathbf{1}_8,\\
D^2A^2 \text{ term}: &\quad \hat{x}_i\Sigma_{ij}\Sigma_{kj}e_k^\dagger\hat{x}^\dagger + \hat{x}_k\Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Sigma_{kj} = & 2\cdot 144\mathbf{1}_8,\\
DA(EA-FB) \text{ term}: &\quad \hat{x}_i\hat{x}_k\Sigma_{ij}\Theta_{kj}\hat{x}^\dagger + \hat{x}_i\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Sigma_{kj} = & -2\cdot 12\mathbf{1}_8,\\
-DA(EA+FC) \text{ term}: &\quad \hat{x}_i\Sigma_{ij}\Theta_{kj}e_k^\dagger + \hat{x}_k\Theta_{ij}e_i^\dagger\Sigma_{kj} = & -2\cdot 12\mathbf{1}_8,\\
D^2B^2 \text{ term}: &\quad \hat{x}_i\hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Sigma_{kj}\hat{x}^\dagger = & -24\mathbf{1}_8,\\
D^2BC \text{ term}: &\quad \hat{x}_i\Sigma_{ij}\hat{x}^\dagger\Sigma_{kj}e_k^\dagger + \hat{x}_k\Sigma_{ij}e_i^\dagger\Sigma_{kj}\hat{x}^\dagger = & -2\cdot 144\mathbf{1}_8,\\
DB(EB+FA) \text{ term}: &\quad \hat{x}_i\hat{x}_k\Sigma_{ij}\hat{x}^\dagger\Theta_{kj} + \hat{x}_i\hat{x}_k\Theta_{ij}\Sigma_{kj}\hat{x}^\dagger = & 2\cdot 12\mathbf{1}_8,\\
-DB(EC-FA) \text{ term.}: &\quad \hat{x}_i\Sigma_{ij}\hat{x}^\dagger\Theta_{kj}\hat{x}^\dagger e_k^\dagger + \hat{x}_k\Theta_{ij}\hat{x}^\dagger e_i^\dagger\Sigma_{kj}\hat{x}^\dagger = & -2\cdot 12\mathbf{1}_8,\\
D^2C^2 \text{ term}: &\quad \Sigma_{ij}e_i^\dagger\Sigma_{kj}e_k^\dagger = & -1008\mathbf{1}_8,\\
DC(EB+FA) \text{ term}: &\quad \hat{x}_k\Sigma_{ij}e_i^\dagger\Theta_{kj} + \hat{x}_i\Theta_{ij}\Sigma_{kj}e_k^\dagger = & 2\cdot 72\mathbf{1}_8,\\
-DC(EC-FA) \text{ term}: &\quad \Sigma_{ij}e_i^\dagger\Theta_{kj}\hat{x}^\dagger e_k^\dagger + \Theta_{ij}\hat{x}^\dagger e_i^\dagger\Sigma_{kj}e_k^\dagger = & -2\cdot 144\mathbf{1}_8,\\
A^2D^2 \text{ term}: &\quad \Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Sigma_{kj}e_k^\dagger\hat{x}^\dagger = & -720\mathbf{1}_8,\\
AD(EA-FB) \text{ term}: &\quad \hat{x}_k\Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Theta_{kj}\hat{x}^\dagger + \hat{x}_i\Theta_{ij}\hat{x}^\dagger\Sigma_{kj}e_k^\dagger\hat{x}^\dagger = & 2\cdot 72\mathbf{1}_8,\\
-AD(EA+FC) \text{ term}: &\quad \Sigma_{ij}e_i^\dagger\hat{x}^\dagger\Theta_{kj}e_k^\dagger + \Theta_{ij}e_i^\dagger\Sigma_{kj}e_k^\dagger\hat{x}^\dagger = & 2\cdot 144\mathbf{1}_8,\\
(EA-FB)^2 \text{ term}: &\quad \hat{x}_i\hat{x}_k\Theta_{ij}\hat{x}^\dagger\Theta_{kj}\hat{x}^\dagger = & -6\mathbf{1}_8,\\
-(EA-FB)(EA+FC) \text{ term}: &\quad \hat{x}_i\Theta_{ij}\hat{x}^\dagger\Theta_{kj}e_k^\dagger + \hat{x}_k\Theta_{ij}e_i^\dagger\Theta_{kj}\hat{x}^\dagger = & -2\cdot 6\mathbf{1}_8,\\
(EB+FA)^2 \text{ term}: &\quad \hat{x}_i\hat{x}_k\Theta_{ij}\Theta_{kj} = & -6\mathbf{1}_8,\\
-(EB+FA)(EC-FA) \text{ term}: &\quad \hat{x}_i\Theta_{ij}\Theta_{kj}\hat{x}^\dagger e_k^\dagger + \hat{x}_k\Theta_{ij}\hat{x}^\dagger e_i^\dagger\Theta_{kj} = & 2\cdot 6\mathbf{1}_8,\\
(EC-FA)^2 \text{ term}: &\quad \Theta_{ij}\hat{x}^\dagger e_i^\dagger\Theta_{kj}\hat{x}^\dagger e_k^\dagger = & -42\mathbf{1}_8,\\
(EA+FC)^2 \text{ term}: &\quad \Theta_{ij}e_i^\dagger\Theta_{kj}e_k^\dagger = & 30\mathbf{1}_8.
\end{aligned} \tag{4.86}
$$

Hence

$$
\mathrm{Tr}[R_i,R_j]R_i[R_k,R_j]R_k = 192r^{-2}\sin^2 f\left(-25r^{-4}\sin^4 f - 16r^{-2}\sin^2(\partial_r f)^2 - (\partial_r f)^4\right). \tag{4.87}
$$

Therefore the results are

$$
\begin{aligned}
\mathrm{Tr}\left([R_i,R_j]^2\right)^2 &= 4608r^{-4}\sin^4 f\left(25r^{-4}\sin^4 f + 20r^{-2}\sin^2 f(\partial_r f)^2 + 4(\partial_r f)^4\right),\\
\mathrm{Tr}\left([R_i,R_j][R_k,R_l]\right)^2 &= 1536r^{-4}\sin^4 f\left(-5r^{-4}\sin^4 f + 20r^{-2}\sin^2 f(\partial_r f)^2 - 8(\partial_r f)^4\right),\\
\mathrm{Tr}[R_i,R_j][R_k,R_l][R_i,R_k][R_j,R_l] &= 768r^{-4}\sin^4 f\left(-55r^{-4}\sin^4 f - 80r^{-2}\sin^2 f(\partial_r f)^2 + 2(\partial_r f)^4\right),\\
\mathrm{Tr}\left([R_i,R_j]\right)^2 R_k^2 &= 192r^{-2}\sin^2 f\left(30r^{-4}\sin^4 f + 17r^{-2}\sin^2 f(\partial_r f)^2 + 2(\partial_r f)^4\right),\\
\mathrm{Tr}\left([R_i,R_j]R_k\right)^2 &= 192r^{-2}\sin^2 f\left(10r^{-4}\sin^4 f + 13r^{-2}\sin^2 f(\partial_r f)^2 - 2(\partial_r f)^4\right),\\
\mathrm{Tr}[R_i,R_j]R_k[R_i,R_k]R_j &= 192r^{-2}\sin^2 f\left(-25r^{-4}\sin^4 f - 16r^{-2}\sin^2(\partial_r f)^2 - (\partial_r f)^4\right),\\
\mathrm{Tr}[R_i,R_j][R_k,R_i]R_jR_k &= 192r^{-2}\sin^2 f\left(15r^{-4}\sin^4 f + 14r^{-2}\sin^2 f(\partial_r f)^2 - (\partial_r f)^4\right),\\
\mathrm{Tr}[R_i,R_j]R_i[R_k,R_j]R_k &= 192r^{-2}\sin^2 f\left(-25r^{-4}\sin^4 f - 16r^{-2}\sin^2(\partial_r f)^2 - (\partial_r f)^4\right).
\end{aligned} \tag{4.88}
$$

Collecting everything altogether, we finally obtain

$$
\begin{aligned}
&\mathrm{Tr}\Big[c_2\left([R_i,R_j][R_i,R_j]\right)^2 + c_2\left([R_i,R_j][R_k,R_l]\right)^2 - 4c_2[R_i,R_j][R_k,R_l][R_i,R_k][R_j,R_l]\\
&\quad + 4c_1\left([R_i,R_j]\right)^2 R_k^2 + 4c_1\left([R_i,R_j]R_k\right)^2 - 4c_1[R_i,R_j]R_k[R_i,R_k]R_j\\
&\quad + 8c_1[R_i,R_j][R_k,R_i]R_jR_k - 4c_1[R_i,R_j]R_i[R_k,R_j]R_k\Big]\\
&= 23040\left(3c_1r^{-4}\sin^4 f(\partial_r f)^2 + 4r^{-6}\sin^6 f\left(4c_2(\partial_r f)^2 + c_1\right) + 12c_2r^{-8}\sin^8 f\right).
\end{aligned} \tag{4.89}
$$

	$\tilde{x}$	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
	$a = 1/3$	0.0100	0.1036	0.2154	0.3379	0.4743	0.6300	0.8143	1.046	1.368	1.939	4.595
	$a = 1/2$	0.0100	0.1054	0.2236	0.3586	0.5164	0.7071	0.9487	1.278	1.789	2.846	9.9
x :	$a = 1$	0.0101	0.1111	0.25	0.4286	0.6667	1	1.5	2.333	4.	9.	99
	$a = 2$	0.0102	0.1235	0.3125	0.6122	1.1111	2.	3.75	7.777	20.	90.	9900.
	$a = 3$	0.0103	0.1372	0.3906	0.8746	1.8519	4.	9.375	25.93	100.	900.	9.9×10^5

Table 4.2: mesh scale of the fractional type ($b = 1$): x vs. $\tilde{x}$

Taking $c_1 = c_2 = 1$ and introducing the overall factor $\int_{S^6} d\Omega_6 = \frac{16}{15}\pi^3 r^6$, we obtain the energy functional (4.16).

4.5 The numerical analysis

In this section, we will discuss numerical calculations to solve the hedgehog equations.

In numerical calculations, we can not directly treat the (semi-)infinite region $x \in [0, \infty]$ because of the region that can be calculating is finite. Hence we have to transform infinite region x to finite region $\tilde{x}$, such as $x \mapsto \tilde{x} \in [0, 1]$. Various types of scale transformation are known for this purpose and we need choose suitable scale transformation for each problem. We usually use the fractional type scale transformation:

$$x = b^{-1} \frac{\tilde{x}}{(1 - \tilde{x})^a} \tag{4.90}$$

where $a \in \{1/4, 1/3, 1/2, 1, 2, 3, 4\}$ is a non-linear transform parameter and $b \in \mathbb{R}$ is linear transform patameter. This scale transformation holds more informative around origin than infinity, thus it is enough that we use this type in usually. Now "more informative" means that mesh points are closer. The reason that we now restrict $a \in \{1/4, 1/3, 1/2, 1, 2, 3, 4\}$ is that there is not the formula of algebraic equation solutions of degree more than five in generally. However it is hard that we calculate the case of $a = 1/4$ or 4, thus we omit these case in this paper. Table 4.2 denotes the scale relations between x with $\tilde{x}$ in the fractional type ($b = 1$). Although we have to take the suitable non-linear transform paramter a for each case, we can change the scale with changing the linear transform parameter b. Thus we usually take the most simple case $a = 1$ and adjust the scale with changing b.

Using the chain rule, the transformation of derivative term (order less than three) becomes

$$\partial_x = \partial_x \tilde{x} \cdot \partial_{\tilde{x}}, \tag{4.91a}$$

$$\begin{aligned}\partial_x^2 &= \partial_x \left(\partial_x \tilde{x} \cdot \partial_{\tilde{x}}\right) \\ &= \partial_x^2 \tilde{x} \cdot \partial_{\tilde{x}} + (\partial_x \tilde{x})^2 \cdot \partial_{\tilde{x}}^2.\end{aligned} \tag{4.91b}$$

For this reason, we need a representation $\tilde{x}(x)$. The inverse functions of the scale relation (4.90) for each a are

$$a = 1/3 \Rightarrow \tilde{x} = \frac{-2 \cdot 3^{1/3} b^2 x^2 + 2^{1/3} bx \left(9 + \sqrt{3(27 + 4b^3 x^3)}\right)^{2/3}}{6^{2/3} \left(9 + \sqrt{3(27 + 4b^3 x^3)}\right)^{1/3}}, \tag{4.92a}$$

$$a = 1/2 \Rightarrow \tilde{x} = \frac{-b^2 x^2 + bx\sqrt{4 + b^2 x^2}}{2}, \tag{4.92b}$$

$$a = 1 \Rightarrow \tilde{x} = \frac{bx}{1 + bx}, \tag{4.92c}$$

$$a = 2 \Rightarrow \tilde{x} = \frac{1 + 2bx - \sqrt{1 + 4bx}}{2bx}, \tag{4.92d}$$

$$a = 3 \Rightarrow \tilde{x} = 1 - \left(\frac{2}{3\left(-9b^2 x^2 + \sqrt{3b^3 x^3 (4 + 27bx)}\right)}\right)^{1/3} + \frac{\left(-9b^2 x^2 + \sqrt{3b^3 x^3 (4 + 27bx)}\right)^{1/3}}{18^{1/3} bx}. \tag{4.92e}$$

We now consider rescaling the Hedgehog equations with using the case of $a = 1$ (and more generalization):

$$[0, \infty] \ni r \mapsto \tilde{r} \in [0, x_{\max}], \quad s.t. \quad r = b^{-1} \frac{\tilde{r}}{x_{\max} - \tilde{r}}, \tag{4.93}$$

then the derivative terms becomes

$$\partial_r = \partial_r\tilde{r}\cdot\partial_{\tilde{r}} = \beta(x_{\max}-\tilde{r})^2\cdot\partial_{\tilde{r}}, \tag{4.94a}$$

$$\partial_r^2 = \partial_r^2\tilde{r}\cdot\partial_{\tilde{r}} + (\partial_r\tilde{r})^2\cdot\partial_{\tilde{r}}^2 = \beta^2(x_{\max}-\tilde{r})^3\left(-2\partial_{\tilde{r}} + (x_{\max}-\tilde{r})\cdot\partial_{\tilde{r}}^2\right), \tag{4.94b}$$

where $\beta \coloneqq b/x_{\max}$.

Using these results, the Hedgehog equation in four dimensions becomes

$$\begin{aligned}
&(r^2+2\sin^2 f)\partial_r^2 f + 2r\partial_r f + \sin 2f\left((\partial_r f)^2 - 1 - \frac{\sin^2 f}{r^2}\right)\\
&= \beta^2\left(b^{-2}\tilde{r}^2 + 2\sin^2 f(x_{\max}-\tilde{r})^2\right)(x_{\max}-\tilde{r})^2\partial_{\tilde{r}}^2 f + 2\beta^2\left(b^{-2}\tilde{r} - 2\sin^2 f(x_{\max}-\tilde{r})\right)(x_{\max}-\tilde{r})^2\partial_r f\\
&\qquad + \sin 2f\left(\beta^2(x_{\max}-\tilde{r})^4(\partial_{\tilde{r}} f)^2 - 1 - b^2(x_{\max}-\tilde{r})^2\frac{\sin^2 f}{\tilde{r}^2}\right).
\end{aligned} \tag{4.95}$$

In numerically calculation, dividing by zero is not good hence we define the normalized functional $F(f,f',f'')$ as multiplying $\tilde{r}^2$ on both side in the equation:

$$\begin{aligned}
F(f,f',f'') &= \beta^2\tilde{r}^2\left(b^{-2}\tilde{r}^2 + 2\sin^2 f(x_{\max}-\tilde{r})^2\right)(x_{\max}-\tilde{r})^2\partial_{\tilde{r}}^2 f + 2\beta^2\tilde{r}^2\left(b^{-2}\tilde{r} - 2\sin^2 f(x_{\max}-\tilde{r})\right)(x_{\max}-\tilde{r})^2\partial_r f\\
&\qquad + \sin 2f\left(\beta^2\tilde{r}^2(x_{\max}-\tilde{r})^4(\partial_{\tilde{r}} f)^2 - \tilde{r}^2 - b^2(x_{\max}-\tilde{r})^2\sin^2 f\right).
\end{aligned} \tag{4.96}$$

Similarly, the Hedgehog equation in eight dimensions becomes

$$\begin{aligned}
&\sin^2 f\left(3r^2 + 16\sin^2 f\right)\partial_r^2 f + 6r\sin^2 f\partial_r f + 3\sin 2f\left(\left(r^2+8\sin^2 f\right)(\partial_r f)^2 - 2\sin^2 f - 8\frac{\sin^4 f}{r^2}\right)\\
&= \beta^2\sin^2 f\left(3b^{-2}\tilde{r}^2 + 16\sin^2 f(x_{\max}-\tilde{r})^2\right)(x_{\max}-\tilde{r})^2\partial_{\tilde{r}}^2 f + 2\beta^2\sin^2 f\left(3b^{-2}\tilde{r} - 16\sin^2 f(x_{\max}-\tilde{r})\right)(x_{\max}-\tilde{r})^2\partial_r f\\
&\quad + 3\sin 2f\left(\beta^2\left(b^{-2}\tilde{r}^2 + 8\sin^2 f(x_{\max}-\tilde{r})^2\right)(x_{\max}-\tilde{r})^2(\partial_{\tilde{r}} f)^2 - 2\sin^2 f - 8\sin^4 f\frac{b^2(x_{\max}-\tilde{r})^2}{\tilde{r}^2}\right).
\end{aligned} \tag{4.97}$$

We define the normalization functional $F(f,f',f'')$ as multiplying $\tilde{r}^2\sin^{-1} f$ on both side in the equation:

$$\begin{aligned}
F(f,f',f'') = {}&\beta^2\sin f\left(3b^{-2}\tilde{r}^2 + 16\sin^2 f(x_{\max}-\tilde{r})^2\right)\tilde{r}^2(x_{\max}-\tilde{r})^2\partial_{\tilde{r}}^2 f\\
&+ 2\beta^2\sin f\left(3b^{-2}\tilde{r} - 16\sin^2 f(x_{\max}-\tilde{r})\right)\tilde{r}^2(x_{\max}-\tilde{r})^2\partial_r f\\
&+ 6\cos f\left(\beta^2\left(b^{-2}\tilde{r}^2 + 8\sin^2 f(x_{\max}-\tilde{r})^2\right)\tilde{r}^2(x_{\max}-\tilde{r})^2(\partial_{\tilde{r}} f)^2 - 2\sin^2 f\tilde{r}^2 - 8b^2\sin^4 f(x_{\max}-\tilde{r})^2\right).
\end{aligned} \tag{4.98}$$

Of course, the boundary condition becomes $f(0)=\pi,\ f(x_{\max})=0$.

The Hedgehog equations are second ordinary differential equation with Neumann boundary condition, thus we solve the equations with using an appropriate numerical method of boundary value problems, such as a shooting method. In this paper, we used a functional Newton-Raphson method (please see the details below).

We now calculate the functional derivative (lower order than two) which is used at the functional Newton-Raphson method. In four dimensions, the functional derivatives are

$$\begin{aligned}
\frac{\partial F(f,f',f'')}{\partial f} = {}&2\beta^2 b^{-2}\sin 2f\tilde{r}^2(x_{\max}-\tilde{r})^4\partial_{\tilde{r}}^2 f - 4\beta^2\sin 2f\tilde{r}^2(x_{\max}-\tilde{r})^3\partial_r f\\
&+ 2\cos 2f\left(\beta^2\tilde{r}^2(x_{\max}-\tilde{r})^4(\partial_{\tilde{r}} f)^2 - \tilde{r}^2 - b^2(x_{\max}-\tilde{r})^2\sin^2 f\right) - b^2(\sin 2f)^2(x_{\max}-\tilde{r})^2,
\end{aligned} \tag{4.99a}$$

$$\frac{\partial F(f,f',f'')}{\partial(\partial_{\tilde{r}} f)} = 2\beta^2\tilde{r}^2\left(b^{-2}\tilde{r} - 2\sin^2 f(x_{\max}-\tilde{r})\right)(x_{\max}-\tilde{r})^2 + 2\beta^2\sin 2f\tilde{r}^2(x_{\max}-\tilde{r})^4\partial_{\tilde{r}} f, \tag{4.99b}$$

$$\frac{\partial F(f,f',f'')}{\partial(\partial_{\tilde{r}}^2 f)} = \beta^2\tilde{r}^2\left(b^{-2}\tilde{r}^2 + 2\sin^2 f(x_{\max}-\tilde{r})^2\right)(x_{\max}-\tilde{r})^2. \tag{4.99c}$$

In eight dimensions, the functional derivatives are

$$\begin{aligned}\frac{\partial F(f,f',f'')}{\partial f} &= \beta^2\left(3b^{-2}\cos f\,\tilde r^2+24\sin f\sin 2f(x_{\max}-\tilde r)^2\right)\tilde r^2(x_{\max}-\tilde r)^2\partial_{\tilde r}^2 f\\ &\quad+2\beta^2\left(3b^{-2}\cos f\,\tilde r-24\sin f\sin 2f(x_{\max}-\tilde r)\right)\tilde r^2(x_{\max}-\tilde r)^2\partial_{\tilde r} f\\ &\quad-6\sin f\left(\beta^2\left(b^{-2}\tilde r^2+8\sin^2 f(x_{\max}-\tilde r)^2\right)\tilde r^2(x_{\max}-\tilde r)^2(\partial_{\tilde r} f)^2-2\sin^2 f\tilde r^2-8b^2\sin^4 f(x_{\max}-\tilde r)^2\right)\\ &\quad+12\cos f\sin 2f\left(4\beta^2\tilde r^2(x_{\max}-\tilde r)^4(\partial_{\tilde r} f)^2-\tilde r^2-8b^2\sin^2 f(x_{\max}-r)^2\right),\end{aligned} \tag{4.100a}$$

$$\begin{aligned}\frac{\partial F(f,f'f'')}{\partial f'} &= 2\beta^2\sin f\left(3b^{-2}\tilde r-16\sin^2 f(x_{\max}-\tilde r)\right)\tilde r^2(x_{\max}-\tilde r)^2\\ &\quad+12\beta^2\cos f\left(b^{-2}\tilde r^2+8\sin^2 f(x_{\max}-\tilde r)^2\right)\tilde r^2(x_{\max}-\tilde r)^2\partial_{\tilde r} f,\end{aligned} \tag{4.100b}$$

$$\frac{\partial F(f,f'f'')}{\partial f''} = \beta^2\sin f\left(3b^{-2}\tilde r^2+16\sin^2 f(x_{\max}-\tilde r)^2\right)\tilde r^2(x_{\max}-\tilde r)^2. \tag{4.100c}$$

4.5.1 Functional Newton-Raphson method

The functional Newton-Raphson method is able to regard as an applied method of the well-known Newton-Raphson method, which find root of function, to solve differential equations. In various context, this method is simply called as the Newton-Raphson method but we call the solving method of differential equations as the functional Newton-Raphson method to distinguish these two method. Although we can solve the partial differential equations with using the functional Newton-Raphson method also, we focus our discussion on the ordinary differential equations case.

The functional Newton-Raphson method is one of the iterative methods which reduce residuals step by step. Roughly speaking, the residual means that an error of functional values on current step. This method can be used for any implicit functions, thus it is useful when we solve an non-linear systems. Although the Newton-Raphson method can be applied any nth order differential equations, for simplify, we restrict the following discussion to the second order ordinary differential equations:

$$F(f(x),f'(x),f''(x))=0, \tag{4.101}$$

Let x be a variable and define the closed interval $x\in[x_{\text{ini}},x_{\text{fin}}]$. Divide this interval into N equal parts. Now we define some values which are used below discussions;

$$\begin{aligned}&\text{mesh size}: && \delta x:=(x_{\text{fin}}-x_{\text{ini}})/N, && \text{mesh point}: && x_i:=x_{\text{ini}}+i\cdot\delta x,\\ &\text{function value on the mesh point}: && f_i:=f(x_i), && \text{functional on the mesh point}: && F_i:=F(f_i,f_i',f_i'').\end{aligned} \tag{4.102}$$

where $i=0,\ldots,N$. In the iterative method, we have to give a reasonable starting value of function, a $f_i^{(0)}$ denote this starting value. Of course, this starting value $f_i^{(0)}$ does not satisfy the functional equation:

$$F\left(f_i^{(0)},f_i'^{(0)},f_i''^{(0)}\right)\neq 0, \tag{4.103}$$

where $f_i'^{(0)}:=\partial_x f^{(0)}(x)|_{x=x_i}$, $f_i''^{(0)}:=\partial_x^2 f^{(0)}(x)|_{x=x_i}$. Suppose that the functional equation is satisfied by using a correction value $\delta f_i^{(0)}$:

$$F\left(f_i^{(0)}+\delta f_i^{(0)},f_i'^{(0)}+\delta f_i'^{(0)},f_i''^{(0)}+\delta f_i''^{(0)}\right)=0. \tag{4.104}$$

We call the correction value $\delta f_i^{(0)}$ as the residual. Assume that each residual $\delta f_i^{(0)},\delta f_i'^{(0)},\delta f_i''^{(0)}$ in (4.104) is not large, and then the Taylor expansion with regarding f_i,f_i',f_i'' as independent values becomes

$$F\left(f_i^{(0)},f_i'^{(0)},f_i''^{(0)}\right)+\left(\frac{\partial F}{\partial f}\right)_i^{(0)}\delta f_i^{(0)}+\left(\frac{\partial F}{\partial f'}\right)_i^{(0)}\delta f_i'^{(0)}+\left(\frac{\partial F}{\partial f''}\right)_i^{(0)}\delta f_i''^{(0)}+O\left((\delta f_i)^2\right)=0. \tag{4.105}$$

Here the symbol $\left(\frac{\partial F}{\partial f}\right)_i^{(0)}$ means $\partial_f F(f,f',f'')\big|_{(f(x)=f_i^{(0)},f'(x)=f_i'^{(0)},f''(x)=f_i''^{(0)})}$ and the others are similar meanings. Now we ignore residuals more than second order, and then rewrite the equation including only the residual on each mesh point $\delta f_{i-1},\delta f_i,\delta f_{i+1}$ by replacing the differential $\delta f_i'^{(0)},\delta f_i''^{(0)}$ with a finite difference of residual (see this below). We take this operation on whole mesh point $i = 0, \ldots, N$ and obtain a linear system of $N+1$ equations in the $N+1$ residual $\delta f_i^{(0)}$ ($i = 0, \ldots, N$), call this system as the residual equations. The coefficients of this system has only $F^{(0)}, \partial_f F_i|^{(0)}, \partial_{f'} F_i|^{(0)}, \partial_{f''} F_i|^{(0)}$, hence the residuals $\delta f_i^{(0)}$ are uniquely decided when we give the starting value $f_i^{(0)}$. We obtain a corrected function value $f_i^{(1)}$ by correcting this residual:

$$f_i^{(0)} + \delta f_i^{(0)} \to f_i^{(1)}. \tag{4.106}$$

This new function value $f_i^{(1)}$ is nearing the truth solution than previous function value $f_i^{(0)}$. Therefore we can calculate a next residual $\delta f_i^{(1)}$ by peforming same calculations, and finally obtain the truth solution f_i to the equation (4.101) by calculating this correcting again and again until the residual $\delta f_i^{(m)}$ is enough small. The algorithm like this is called as the functional Newton-Raphson method.

In above discussion, still we do not give the residual equations explicitly, let us give these equations in the following. The explicit form of the residual equations is different with the type of boundary conditions, namely the boundary problems (i.e. the Neumann boundary condition) or the initial problems (i.e. the Dirichlet boundary conditions), and the order precision of using finite difference. In the boundary problems then the function value at the both side $i = 0, N$ is fixed, thus the functional Newton-Raphson method correct the function value in the inner $i = 1, \ldots, N-1$. On the other hand, in the initial prolems, we fix the function value at the initial point $i = 0$ and correct the function value in the other points $i = 1, \ldots, N$. Moreover, because that the number of using mesh points by the finite difference is changed with the order precision of using finite difference, we have to use the forward or backward finite difference at suitable points around the both side. We now use the second order precision finite difference and consider the boundary problems. For simplify, in below discussion, we omit the index that is meant step numbers such that $^{(0)}$.

The second order presicion central finite difference is given by

$$\delta f_i' = \frac{f_{i+1} - f_{i-1}}{2\delta x}, \quad \delta f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\delta x)^2}. \tag{4.107}$$

We now consider the boundary problems thus the region of calculations becomes $i = 1, \ldots, N-1$. Hence it is enough that we use the above central finite difference when using the second order precision finite difference. Using (4.107), the expansion equation (4.105) with ignore higher order residuals becomes

$$\left[\frac{1}{2\delta x}\left(\frac{\partial F}{\partial f'}\right)_i - \frac{1}{(\delta x)^2}\left(\frac{\partial F}{\partial f''}\right)_i\right]\delta f_{i-1} + \left[-\left(\frac{\partial F}{\partial f}\right)_i + \frac{2}{(\delta x)^2}\left(\frac{\partial F}{\partial f''}\right)_i\right]\delta f_i + \left[-\frac{1}{2\delta x}\left(\frac{\partial F}{\partial f'}\right)_i - \frac{1}{(\delta x)^2}\left(\frac{\partial F}{\partial f''}\right)_i\right]\delta f_{i+1} = F_i, \tag{4.108}$$

where i run from 1 to $N-1$. We note that $\delta f_0 = \delta f_N = 0$ because of the boundary problems, and we obtain the residual equation by collecting the above $N-2$ equations:

$$\begin{pmatrix} B_1 & C_1 & 0 & \cdots & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_{N-2} & B_{N-2} & C_{N-2} \\ 0 & \cdots & 0 & A_{N-1} & B_{N-1} \end{pmatrix}\begin{pmatrix} \delta f_1 \\ \delta f_2 \\ \vdots \\ \delta f_{N-2} \\ \delta f_{N-1} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-2} \\ F_{N-1} \end{pmatrix}, \tag{4.109}$$

where

$$A_i := \frac{1}{2\delta x}\left(\frac{\partial F}{\partial f'}\right)_i - \frac{1}{(\delta x)^2}\left(\frac{\partial F}{\partial f''}\right)_i, \qquad B_i := -\left(\frac{\partial F}{\partial f}\right)_i + \frac{2}{(\delta x)^2}\left(\frac{\partial F}{\partial f''}\right)_i, \qquad C_i := -\frac{1}{2\delta x}\left(\frac{\partial F}{\partial f'}\right)_i - \frac{1}{(\delta x)^2}\left(\frac{\partial F}{\partial f''}\right)_i. \tag{4.110}$$

This equation is the system of $N-2$ linear equations, thus we can numerically sovle the residuals δf_i ($i = 1, \ldots, N-1$) with using an appropriate algorithm of matrix calculations. Using these residuals, we correct the function value step by step. At finally the residual becomes enough small.

Some comments are in order. First, the residuals around begining step when the function value differ from solution are sometimes big values and it is not so easy to converge one. Hence we use the modified correction equation which is multiplied the appropriate boost parameter ϵ around begining step:

$$f_i + \epsilon\delta f_i \to f_i, \tag{4.111}$$

for example we take $\epsilon = 0.01, 0.1$ etc. then the residual more earlier converge than using original one (4.106).

Second, the matrix of the residual equation (4.109) is the band matrix thus the calculation can speed up with using the appropriate numerical package (for example “LAPACK”).

Conclusion

In this paper we have studied mainly two topics: the one is about the ADHM construction and the (self-dual type) instantons in 4n dimensions, the other is about the Atiyah-Manton construction and Skyrmions in eight dimensions.

It is well known that the usually four-dimensional ADHM construction is based on the quaternion. Naturally, it is expect that a algeraic basis which is generalization of the quarternion will palys important role in the higher dimensional ADHM construction. Indeed, we have shown that the framework of the $4n$-dimensional ADHM construction was construced by using the ASD basis which is constructed from the $(4n-1)$-dimensional Clifford algebra. Moreover we have proved that the $4n$-dimensional ASD tensor, which is generalization of the 't Hooft symbol in four dimensions, from the ASD basis satisies the Hodge duality equation in $4n$ dimensions. The Hodge duality equation is more generalization of the ASD equation. The Clifford algebra is one of generalization algebras of the quaternion, thus this scheme is the straightforward generalization of the usually ADHM construcion. The (self-dual type) instanton in $4n$ dimensions is defined as solution to the $4n$-dimensional ASD equations. Compared with the ASD equation in four dimensions, the equation in $4n$ dimensions is non-linear. We have found that there are two ADHM constraints which is duality equation the ASD equation, one of these is the straightforward generalization of four-dimensional one. The other ADHM constraint, which is the new constraint, corresponds to the non-linearity of the higher-dimensional ASD equation. One of the most interesting things is that the more non-linearity of the ASD equations is according as dimensions increase but the ADHM construction does not need essentially new constraints more than what is shown in this paper. We have shown that our construction reproduces the known BPST one instanton in 4n dimensions by using the simplest generalization ADHM data of four-dimensional one. On the other hand, unfortunately, we have found that the straightforward generalization of the well-known four-dimensional multi-instanton ADHM data, namely the 't Hooft type data and the JNR type data, do not satisfy the second ADHM constraint in generally. These multi-instanton ADHM data become well defined if only if the data satisy the well-separated limit. This fact means that the construction of the higher dimensional multi-instantons is not easy. Therefore one of the futerworks is finding the suitable multi-instantons ADHM data in higher dimesions.

In the latter part of this paper, we discussed closed connection between the instantons and the Skyrmions in the higher dimensions. The Skyrmions in four dimensions are solutions to the (static) Skyrme model, thus we need the higher dimensional Skyrme model to consider the higher dimensional Skyrmions. Following the formalism developed by Sutcliffe truncation method, we have shown that the static Skyrme model can be derived from the generalized Yang-Mills model in $4n$ dimensions. This higher dimensional Skyrme model satisfy the Derrick's theorem, thus we expext stable soliton solutions in this model and call these solutions as the higher dimensional Skyrmions. In particular, we have considered the eight dimensional Skyrme model and shown that the lower bound of this action is the topological charge which is the generalization of the Baryon number. Furthermore we have introduced the higher dimensional spherically symmetric ansatz, namely the higher dimensional hedgehog ansatz, and presented the explicit numerical solution for the Skyrmion associated with the topological charge one. The profie function and the energy density of the eight-dimensional single Skyrmion look quite similar to those in four dimensions. In four dimensions, it is known that the Atiyah-Manton construction lead the good approximate profile function of Skyrmions from the instantons. Following the four-dimensional case, we constructed the Atiyah-Manton solution for the Skyrmin from the eight-dimensional one instanton solution. We then compare the above numerical solution and the Atiyah-Manton solution and find that there is a good agreement between them. This result means that the Atiyah-Manton construction works well in eight dimensions, and dictates us that the correspondence the instantons and the Skyrmions by the Atiyah-Manton construction is an universal property in higher dimensions.

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