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Geometric Aspects of Gauge Algebroids and T-duality in String Theory

森 遥 (DS-20902)
指導教授　量子物理学　川崎健夫

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分子科学専攻 量子物理学
DS-20902 森 遥

現代物理において，初期宇宙は非常に小さく，かつ高エネルギーであったとされる．その時空構造を紐解くことは，興味深い問題のひとつである．現在の宇宙の様子，すなわちマクロな時空の構造は一般相対性理論によって記述されるが，この理論には適用限界がある．高エネルギー領域でのミクロな時空を記述するには一般相対性理論の拡張を行い，量子重力理論を構築する必要がある．量子重力理論の最有力候補が超弦理論である．

超弦理論では，プランク長 $(10^{-34}\mathrm{m})$ の弦を用いて時空の構造を調べる．また，超弦理論には無矛盾だが一見すると異なる枠組みが 5 つ存在し，それらはいくつかの双対性を通して結びつく．特に，T 双対性は弦がコンパクトな空間に巻きつくことに由来する，弦理論に特有な双対性である．T 双対性は理論同士の関係を示すものであり，超弦理論ではあらわに見ることができない．近年，双対性に着目した重力理論がいくつか提唱されており，これらは総称して Extended Field Theory (ExFT) と呼ばれる．その中でも，Double Field Theory (DFT) は，T 双対性に着目した新しい重力理論である．DFT の特徴は，通常の物理的な空間を表す x 座標系と，弦の巻き付きに由来する $\tilde{x}$ 座標系のふたつを持った，倍化された座標上で定義されていることである．これは端的に言えば，弦の巻きつきによる効果を考慮する形で一般相対性理論を拡張したものと思ってよい．この座標系のもとで，T 双対性は x と $\tilde{x}$ の入れ替えとして明示される．また，時空の自由度を増やしたことと引き換えに，DFT は拘束条件をもつ．この条件は，倍化された空間から，通常の物理的な空間を選択する意味合いがある．

一般相対性理論が，リーマン幾何学を用いて定式化されたように，時空構造を理解する上で，幾何学と重力理論は密接に関係する．DFT も背景になんらかの幾何学的描像をもっているべきであるが，先に述べたような x 座標と $\tilde{x}$ 座標が共存する DFT の座標系は，リーマン幾何学では記述しきれない．このような空間の幾何学は，倍化幾何学 (doubled geometry) と呼ばれ，para-Hermitian 幾何学や Born 幾何学などと関連している．DFT に対しての倍化幾何学に限らず，一般に ExFT に対する幾何学的な描像が考察されており，これらは extended geometry と呼ばれている．

本論文では，DFT という理論が持つ局所対称性に注目して，DFT の理論構造やその幾何学について議論する．一般的には，ある理論が局所対称性を持つならば，それは数学的には群の代数構造と関連する．ところが，DFT の局所対称性からは，代数構造ではなく，その一般化である亜代数 (algebroid) 構造が現れることが知られている．まずは，亜代数のもつ直和構造について明らかにする．また，DFT の構造から，亜代数の twist と呼ばれる変形についても議論する．これは，量子群の分野でよく知られた，Hopf 代数の Drinfel'd double の拡張として解釈される．また，この構造を倍化された空間の上で具体的に明示する．これより DFT の持つ拘束条件の代数的な起源は，亜代数がもつ直和構造から明らかになることがわかった．

Abstruct

In modern physics, it is known that the search for the fundamental unit of matter leads to elementary particles. In the field of elementary particles, there are four fundamental interactions: the electromagnetism, the weak interaction, the strong interaction, and the gravity. We need a new framework in which gravity and quantum theory coexist to understand physics on a microscopic scale. For example, this framework is needed to consider the early universe or a singularity in a black hole at close range.

Currently, superstring theory is the most promising candidate for a quantum gravity theory. There are five consistent string theories. The five string theories are connected by string duality. While "symmetry" refers to the invariance of the laws of physics within a single theory, "duality" refers to the fact that two seemingly different theories are actually physically equivalent. T-duality is one of the dualities that arise when a string has length and winds around a compactified space. As long as one focuses on a string theory, it is not possible to investigate T-duality explicitly.

Recently, a new gravity theory has been developed which has T-duality as "symmetry". This is called Double Field Theory (DFT). It is defined on a doubled space, which contains not only the ordinary spacetime coordinate x (Fourier conjugate of Kalza-Klein momentum) but also the winding coordinate (Fourier conjugate of the string winding modes). DFT is a T-duality covariant reformulation of the supergravity.

Historically, gravity theory and geometry are closely related. The most famous theory describing classical gravity is Einstein's general relativity. The breakthrough of the general theory of relativity is that it considers mass as a distortion of space. Riemannian geometry played an important role in this theory. Just as general relativity was formulated in Riemannian geometry, DFT (and doubled geometry) are assumed to have some mathematical geometric pictures. However, it also suggests a new geometry that is completely different from Riemannian geometry.

We are interested in the theoretical (mathematical) structure of DFT. DFT has the strong constraint for consistency. We discuss the mathematical origin of the constraint to focus on the gauge symmetry in DFT. There are various interesting mathematical structures that do not appear in ordinary physics.

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1 Introduction

Physical requirements trigger the development of mathematics. Mathematics also plays an important role in the development of physics. For example, Newton introduced differential and integral calculus, in particular higher-order derivatives and Taylor series. These later became the basis of mathematical analysis. The reason for the introduction of these mathematical notions was to describe the classical laws of physics. The fundamental book on classical mechanics known as "The Mathematical Principles of Natural Philosophy (Philosophiæ-Naturalis Principia Mathematica)" was published by Newton in 1687. In this book, the three laws of motion and universal gravitation are discussed mathematically. Subsequently, differential and integral calculus were systematically established by Leibnitz et al. One of Newton's achievements is the approach of "replacing a natural phenomenon into a mathematical problem and solving it" itself.

One of the most obvious examples of the connection between mathematics and physics is gravity. Gravity is a very familiar force for us. The existence of gravity itself has long been recognized, as the Newton apple anecdote shows. The most famous theory describing classical gravity is probably Einstein's general relativity [1]. The breakthrough of general relativity is that it considers mass as a curved space. Riemannian geometry played an important role in describing this theory. This geometry generalizes the concept of parallelism to curved spaces by introducing the concept of connection. The characteristic of Riemannian geometry is that it gives concrete quantities (or objects) such as curvature and torsion that give space its shape. Thus, gravity theory and geometry are closely related. Another important concept in general relativity is symmetries. This theory is based on the principle of relativity, i.e., that all physical laws are invariant (symmetric) under general coordinate transformations. The general coordinate transformations are represented by the $GL(3,1)$ group, and physical laws are described based on this symmetry.

Today, general relativity is well known as the classical theory describing gravity. It has been corroborated with great precision from an experimental point of view. These include pulsar timing measurements [2], the problem of Mercury's perihelion [3], and more recently, experimental confirmation of gravitational waves [4] and the imaging of black holes [5]. There is no doubt that on the macroscopic scale, the classical theory of general relativity is sufficient.

In modern physics, it is well known that the fundamental objects are elementary particles. There are four types of forces, or interactions for elementary particles: electromagnetism, gravity, weak interaction, and strong interaction. The weak interaction is related to β decay, the radioactive decay of nuclei. The same strong interaction binds protons and neutrons together to form a nucleus. These quantum interactions are described by Quantum Field Theory (QFT). An important aspect of QFT is the existence of the gauge group, which describes the symmetry of elementary particles. In the four interactions, the electromagnetism was the first to be successfully quantized. The quantization of the electromagnetic field was first realized by Dirac in 1927 [6]. Later, Fermi formulated Quantum Electrodynamics (QED) to the first-order of perturbation theory [7] . The success of QED prompted the formulation of other interactions using QFT. Subsequently, the strong and weak interactions were described theoretically by extending the symmetry from Abelian group to non-Abelian group. Glashow-Weinberg-Salam's theory gives a unified description of the weak interaction and the electromagnetism [8–10]. This electroweak unified theory is a gauge theory with gauge group $SU(2) \times U(1)$. This gauge symmetry is spontaneously broken to $U(1)$, the gauge group of QED, by the Higgs mechanism [11–13]. The strong interaction is described by Quantum Chromodynamics (QCD), a non-Abelian gauge theory based on $SU(3)$ symmetry. The basic theoretical structure of QCD was proposed by Yang and Mills in 1954 [14]. Thus, it is clear that group structure and algebraic structure are important concepts governing symmetry even on the microscopic scale.

Glashow-Weinberg-Salam's theory and QCD together are called the Standard Model (SM). On the other hand, the behavior of elementary particles has been experimentally verified using particle colliders. New particles are produced in particle collisions with appropriately high energy at the center of mass. Their properties, such as charge and momentum parameters, are analyzed by several detectors placed around the collision point. From these data, the fundamental interactions between particles governed by QFT can be reconstructed. QFT has been validated with high accuracy at least up to the current 13 TeV energy scale obtained in proton collisions by the LHC. However, the SM is not perfect in describing the interactions of elementary particles, because it does not incorporate gravity. As mentioned at the beginning, general relativity gives a very clear description of gravity on the macroscopic scale. On the other hand, we need other new theories in which gravity and quantum theory coexist to understand physics on a microscopic scale. Currently, superstring theory is considered the most promising candidate for a quantum gravity theory.

1.1 String theory

String theory was originally proposed to describe strong interactions and to classify hadrons [15, 16]. The fundamental object of the theory is not a particle but a string with length. The original feature of this theory was that the modes of vibration of this string represented different hadrons. Nevertheless, there were some problems, such as the dimension of spacetime is 26 and that tachyon appeared as the lightest particles. After the development of QCD in the 1970s, it was recognized that QCD was the correct theory to describe strong interactions, not string theory. However, it was pointed out that the spin-2 excitation in relativistic closed strings is related to gravity [17–19] in 1974. String theory was once again noticed as a possible candidate for a theory of quantum gravity. If we regard string theory as a theory of gravity, then supersymmetry (SUSY) is required to treat fermions in string theory. A string theory with SUSY is called superstring theory. If we consider superstring theory as a theory of gravity, it has the following characteristics.

- Consistency in superstring theory requires that the spacetime dimension is 10.
- Massless spin-2 particle (graviton) appears in the spectrum of the theory.
- Discovery of the anomaly cancellation mechanism that allows gauge group $SO(32)$ and $E_8 \times E_8$. These groups include $SU(3) \times SU(2) \times U(1)$ and are large enough to consider parity breaking (required by electroweak interactions).
- The only artificial parameter is the string tension T. The coupling constant g_s of the dimensionless string is determined by the expectation value of the scalar field (dilaton).
- There are five equivalent theories (type IIA, type IIB, type I, $SO(32)$ heterotic, $E_8 \times E_8$ heterotic).

In this thesis, we will use the term "string theory" simply to refer to superstring theory. String theory without SUSY will be referred to as bosonic string theory.

Interactions in string theory occur in the extended region of spacetime that originates from the length of the string. This is expected to prevent the point-particle divergence inherent in QFT calculations and make string perturbation theory UV-finite. Strings have Planck length $l_{Pl} = \sqrt{\hbar G/c^3} \sim 10^{-35}m$. This is expressed using the fundamental constants in the theory of gravity and quantum theory (In this thesis, we use the natural units, $c = \hbar = 1$).

There are two kinds of strings. One is an open string, which has endpoints like fishing lines. Another is a closed string, which does not have endpoints like rubber bands. The oscillating modes of the string correspond to elementary particles in spacetime. In other words, quantizing a relativistic string yields a generation operator. By applying them to zero mode, states with various masses, i.e. particles, can be generated. The ground state of the string corresponds to a tachyon, but as mentioned above, this mode is eliminated in string theory. The first excited state of an open bosonic string is a photon-like massless vector field. The first excited state of a closed bosonic string corresponds to a massless second-order tensor field containing gravitons.

String theory lives in a region of high energy that experimental observation is impossible. Higher-order excited states in string theory include massive particles. When considering the current universe or macroscopic space, it is common to consider only the massless state, where the length of the string l_s is zero. This is called the low-energy effective theory of

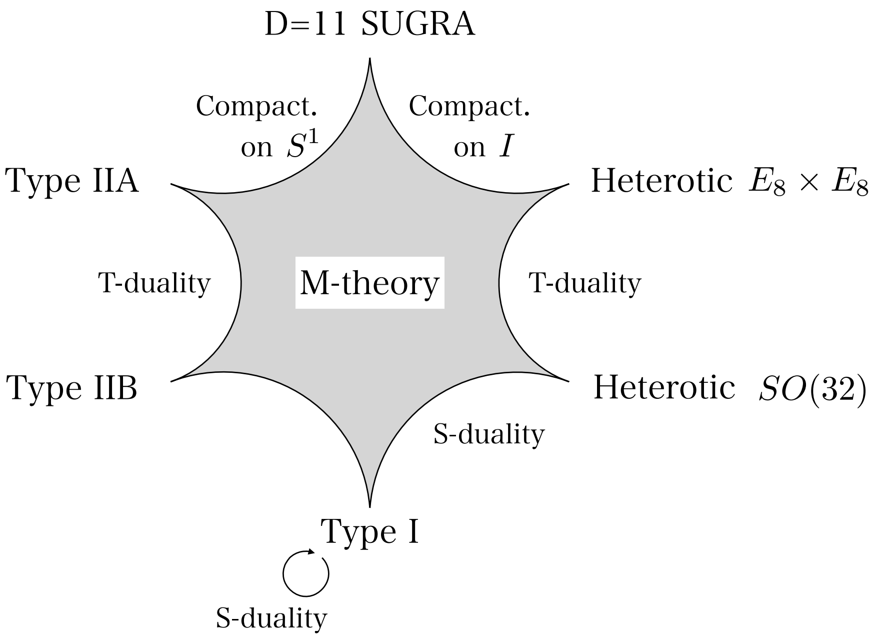

Figure 1.1: The duality chain of string theories.

string theory. The closed string massless state can be divided into the symmetric part, the anti-symmetric part, and the trace part of the second-order tensor. These correspond to the gravity field, the Kalb-Ramond field, and the dilaton field, respectively. The equations of motion for these fields are obtained from the condition that the worldsheet anomaly of conformal symmetry vanishes [20]. The theory giving these equations of motion is called supergravity (SUGRA). Supergravity is an extension of general relativity in a way that requires SUSY. In other words, the low-energy limit of string theory is supergravity.

The dimensions of string theory are too large, whereas our perceived spacetime is four-dimensional. Thus, the question arises as to how to deal with the extra dimensions. One way to deal with the extra dimensions is to curl up the space compact enough not to be detected at low energies. This procedure is called compactification and originally comes from the Kaluza-Klein (KK) theory [21, 22]. If the internal space of six dimensions is appropriately chosen, compactification will yield a four-dimensional string theory.

There are five consistent string theories: type I, type IIA, type IIB, heterotic $SO(32)$, and heterotic $E_8 \times E_8$. The type II and heterotic string theories are only closed string theories, while type I contains both closed and open strings. The type II and heterotic theories share a common bosonic subsector called the Neveu-Schwarz (NS-NS) sector. It contains a metric $g_{\mu\nu}$, an antisymmetric 2-form known as the Kalb-Ramond field $B_{\mu\nu}$, and a scalar known as the dilaton ϕ. Also, type I is given by $SO(32)$ for the gauge group, and in heterotic string theory the type of string (in terms of whether it contains SUSY or not) differs between left mover and right mover. The difference between IIA and IIB is the chirality of the fermion. The five string theories are connected by string duality (fig 1.1). While "symmetry" refers to the invariance of the laws of physics within a theory, "duality" refers to the equivalence of the two seemingly different theories. For example, type I theory with the coupling constant g_s is equivalent to the $SO(32)$ heterotic theory with the coupling constant $1/g_s$. This duality is called S-duality [23]. S-duality also transfers type IIB to type IIB itself. In addition, type IIA string theory is equivalent to type IIB string theory with toroidal compactification [24, 25]. This duality is called T-duality [26]. Heterotic $SO(32)$ string theory and $E_8 \times E_8$ string theory are also related by T-duality. Also, type I is known to be related to type I' by T-duality, although this is not shown in fig 1.1 [25, 27]. The existence of these string dualities suggests that the five string theories are not different theories but describe different aspects of one mother theory. This theory is called M-theory [28–31]. The low-energy limit of M-theory is 11-dimensional supergravity. The U-duality [32], which combines S-duality and T-duality, is expected to be a complete M-theory symmetry.

1.2 T-duality and related geometries

Let us now turn to T-duality. T-duality is the duality that arises when a string has length and winds around a compactified space. Let $\alpha' = l_s^2$ (α' is called slope parameter). It was Kikkawa-Yamasaki [24] and Sakai-Senda [25] who first observed that the spectrums of the strings on a torus are equivalent under the interchange of radius R and α'/R. However, as mentioned earlier, duality is a concept that indicates the relationship between theories. As long as one focuses on a single string theory, it is not possible to investigate T-duality explicitly.

The simplest example where T-duality appears is in a compactified spacetime S^1 only in the one-dimensional direction. Since the string is a one-dimensional object, it winds around the compactified space S^1. Let R be the radius of S^1. The momentum p of the string in the direction of S^1 is $p = m/R$ ($m = 0, \pm 1, \pm 2, \cdots$ is the KK-momentum). On the other hand, energy is also generated by the string wrapping around S^1. This is obtained by the product of the string tension $T_s = (2\pi\alpha')^{-1}$ and the circumference $2\pi R$ (assuming $\alpha' = l_s^2$ and the natural unit system $\hbar = c = 1$). This result contributes to the string mass spectrum M^2 and can be written approximately as follows.

$$M^2 \propto \left(\frac{n}{R}\right)^2 + \left(\frac{mR}{\alpha'}\right)^2, \qquad n, m \in \mathbb{Z} \tag{1.1}$$

where n is an integer that indicates the number of times the string is wound. The first term represents the momentum of the string and the second term represents the energy derived from the tension of the string. The spectrum of this string becomes the same form under the interchange of radius $R \Leftrightarrow \alpha'/R$ as the string momentum m and the winding number n are exchanged. The fact that the spectrum of the theory is invariant under this interchange is called T-duality. It can be read that strings describe spacetime in a different way than point particles. More generally, T-duality can be considered even when the compactification space is geometrically different from S^1. For example, when the compactification space is a D-dimensional torus T^D, then T-duality is extended to the $O(D, D, \mathbb{Z})$ group [33,34]. At least in the low energy limit of supergravity with $\alpha' \to 0$ (zero-slope limit), $O(D, D, \mathbb{R})$ appears as a T-duality group (anomaly breaks $\mathbb{Z}$ into $\mathbb{R}$). Recall that "symmetry" was described by algebra and groups in general relativity and QFT. Similarly, "duality" is essentially governed by these structures. More generally, if we consider string theory (or even supergravity) in a curved target space with isometry, T-duality can be interpreted as a map between different backgrounds. This relation between backgrounds is known as the Buscher rule [35,36]. The Bucher rule is a transformation in which the metric $g_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$ are treated equally and mixed with each other. Mathematically, T-duality ties two completely different kinds of geometry. Just as general relativity was formulated in Riemannian geometry, string theory is assumed to have some geometric picture. However, it seems to suggest a new geometry that is completely different from Riemannian geometry.

A geometry related to T-duality is the generalized geometry proposed by Hitchin [37]. In the generalized geometry, on a manifold M, the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ is introduced. In this geometry, the $O(D, D)$ group, which is the same as the T-duality group, becomes the symmetry group of the target space by introducing $\mathbb{T}M$. This makes it a very useful setup for studying string theory and its low-energy limit, supergravity. Another characteristic feature of this geometry is the appearance of algebroid instead of algebra [38,39]. To put it simply, an algebroid contains information on a manifold (or space-time in physical terms), this is the main difference between an algebroid and algebra [40]. This structure appears as a result of mixing the usual diffeomorphism and B-field gauge transformations and recombining them into an $O(D, D)$-covariant expression.

One of the physical applications of the generalized geometry is related to flux compactification [41]. The algebroid is deformable enough to incorporate a 3-form [42]. This 3-form physically corresponds to the H-flux appearing in the NS-NS sector of type II supergravity. From the conventional T-duality argument based on the Buscher rule, It is known that a T-duality chain (H, F, Q, R) starting from H-flux appears. Here, H, F are called geometric flux, and Q, R are called non-geometric flux. It is pointed out that "non-geometric" flux can be treated geometrically by using the generalized geometry [43–45]. The relationship with the non-linear sigma model has also been studied. The non-linear sigma model is a scalar field theory with nonlinear coupling that describes string theory in a curved background. In its Hamiltonian form, the diffeomorphism and the gauge transformation of the Kalb-Ramond field $B_{\mu\nu}$ are a part of the canonical transformation on the phase space, not a mere field transformation. The canonical transformation corresponds to a general coordinate transformation on the phase space and can be regarded as a geometric symmetry of the phase space. In fact, the generators of the diffeomorphism and the gauge transformation of the B-field in the Hamiltonian form are known to be in one-to-one correspondence with the basic structures of the generalized geometry (generalized tangent bundle and algebroid as a gauge algebra, etc.) [46]. Gates-Hull-Rocek geometry discovered in the study of $2D$-dimentional non-linear sigma models with $\mathcal{N} = (2,2)$ SUSY [47] (it is also called bi-Hermitian geometry) and the generalized Kähler geometry defined in the framework of the generalized geometry are also known to be equivalent [38, 48].

Another geometry related to T-duality is the doubled geometry proposed by Hull [49]. As mentioned earlier, string duality, including T-duality, is a concept that indicates the relationship between theories. Therefore, T-duality is not usually manifested as long as one focuses on a single string theory. Doubled geometry is a new geometry developed under the attempt to treat T-duality explicitly as a "symmetry" (this is called doubled formalism) [50–53]. The geometry of doubled spacetime has been studied mainly by [54–58], in addition to the one by Hull mentioned above. Other relevant discussions are mainly [59–62]. Here, the concept of a duality symmetric sigma model was given, in which the target space is doubled in order to make T-duality a symmetry. In particular, in superspace, the full action of the T-duality symmetry of the low-energy effective theory of superstring theory is given in [51, 52]. As a further development of these theories, there exists a field theory on the doubled geometry, the Double Field Theory, which will be introduced below.

1.3 Double Field Theory

Double Field Theory (DFT) [63] is a field theory defined in the doubled space. DFT is constructed by the discussion in the doubled geometry and String Field Theory (SFT) [64, 65]. The D-dimentional flat SFT of a closed string with torus compactification is discussed in [65–67]. In particular, it was shown in [65] that T-duality is realized as a “symmetry” of the SFT. In SFT, momentum m and winding number n are equivalent. From the Fourier Conjugate, in addition to the usual physical coordinate x for momentum, a new coordinate $\tilde{x}$ corresponding to the number of windings appears. The $\tilde{x}$ is called the winding coordinate. T-duality corresponds to the interchange of momentum m and winding number n, as described above. It appears the interchange of x and $\tilde{x}$ in coordinate space. Therefore, to make T-duality explicit as a “symmetry”, we adopt both x and $\tilde{x}$ as coordinates at the same time, and double the degrees of freedom. That is, in an originally D-dimensional theory, when n-dimensions are toroidal compactified, the theory is constructed on $M \times T^{2n}$. Here, M is the Minkowski spacetime in $(D-n)$ dimension and T^{2n} is the doubled torus $T^n \times T^n$ of the compactified space. Thus, DFT is a field theory that depends on both the normal coordinate x and the winding coordinate $\tilde{x}$.

Based on the above discussion, the currently well-studied DFT is defined on a doubled spacetime in $2D$ dimensions. This is achieved by doubling the all space, not just the compactified space. The value of D is arbitrary, but since the origin of DFT is SFT, $D = 10$ is often assumed. Since the DFT that is currently well studied is based on the type II theory, it contains $g_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$, and the dilaton ϕ. In this sense, it is sometimes called type II DFT to distinguish it from the heterotic DFT described below. The type II

DFT can be said to be a T-duality ($O(D, D, \mathbb{R})$ group) covariant reformulation of the type II supergravity. All these fundamental fields are recombined as T-duality covariant fields on the doubled spacetime. For example, $g_{\mu\nu}$ and $B_{\mu\nu}$ are treated as a unified generalized metric. DFT actions are composed of the generalized metric and the rescaled DFT dilaton. DFT actions are invariant by the T-duality group. DFT action is also invariant by generalized diffeomorphism by a generalized Lie derivative. The generalized Lie derivative is an extension of the Lie derivative in Riemannian geometry. It is generated by vectors defined on doubled spacetime. The generalized Lie derivative is governed by the C-bracket. DFT also has a constraint condition for consistency. This is the closure condition of the gauge algebra which is described by the C-bracket.

The equations of motion of DFT are written in $O(D, D)$-covariant form, which is an extension of the Einstein equation in doubled spacetime. Solutions to DFT are discussed in [68]. In this sense, DFT can also be interpreted as a low-energy effective theory of string theory [69–71]. It should be noted, however, that these formulations are only local. As there is Riemannian geometry for general relativity, a framework of the "global" doubled geometry is needed so that the elements for constructing the DFT (generalized metric, operations such as generalized Lie derivative, or algebroid structure) can be defined globally.

One of the approaches to consider the global formulation of DFT is the para-Hermitian geometry [72–75]. The para-Hermitian structure is the combination of the para-complex structure and the $O(D, D)$ metric. A manifold with this structure is called a para-Hermitian manifold, and it is known that the structure of doubled spacetime appears naturally. Also, as a generalization of the para-Hermitian geometry, Born geometry exists [74, 76]. The born structure is a para-Hermitian structure plus a Riemannian structure $\mathcal{H}$. The sigma model based on Born geometry is discussed in [77, 78]. The relationship between Born geometry and the generalized geometry (generalized Kähler structure) is discussed in [79].

From figure 1.1, we can see that T-duality also exists among heterotic strings. Naturally, it is possible to consider a heterotic theory as "DFT". Heterotic DFT was first formulated by Hohm-Kwak [82], and also discussed in these papers [80, 81]. At present, it is known that ten-dimensional heterotic SUGRA action can be obtained from heterotic DFT action [83]. The relation with the heterotic sigma model is discussed in [84]. In addition to the type II DFT components (metric $g_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$, and the dilaton ϕ), the gauge field ${A_\mu}^a$ is contained in the heterotic DFT, The characteristic point of the heterotic DFT is that the doubled spacetime is extended again by the freedom of this gauge field. The geometry of the heterotic DFT is discussed, for example, in [80]. There is a problem with how to deal (globally) with the space re-extended by the gauge field, which is even more unclear compared to the type II DFT. The geometry of "DFT" is still under development.

As we have mentioned, behind T-duality and DFT, which is a symmetry of the theory, there are hidden various mathematical objects that do not appear in conventional physics. On the other hand, it goes without saying that studying these structures is essential to understanding the world seen by string, i.e., the origin of the universe and the interior of black holes.

1.4 Outline of the thesis

In this thesis, we investigate the gauge algebra of DFT, focusing on the algebroid given by the C-bracket. In particular, we investigate the theoretical structure of DFT in the framework of the para-Hermitian geometry.

The contents of this thesis are based on [84–86]. In particular, a part of Chapter 2 is based on [84], Section 3.4 to Section 3.7 are based on [85, 86], Section 4.2 is based on [85], and the discussion in Chapter 5 is based on [85, 86]. Complementary content other than the above is taken from the publications listed in Bibliography. This thesis is organized as follows.

In Chapter 2, we review Double Field Theory, a field theory covariant with T-duality. First, we introduce the most basic type II DFT action. Next, we consider the generalized Lie derivative and the gauge symmetry of type II DFT. The gauge symmetry is described by the C-bracket. We discuss the section constraints for consistency in DFT. The strong constraint is the closure condition of the gauge algebra described by the C-bracket. An introduction to Heterotic DFT will also be given in this section.

In Chapter 3, we introduce two important concepts for studying gauge algebra in DFT. The first is the concept of Drinfel'd double. The Drinfel'd double is simply the operation of taking a direct sum over two dual Lie algebras. The second is the algebroid structure. The most basic algebroid structure is the Lie algebroid. This is simply an extension of the Lie algebra structure to vector bundles on a manifold M. By preparing a dual vector bundle and introducing two Lie algebroids, Drinfel'd double can be performed on those Lie algebroids under the compatibility condition. This compatibility condition is called the derivation condition. By analogy with Drinfel'd double of Lie algebroid, we show various algebroid structures also have a doubled structure [85, 86]. Especially, the Vaisman algebroid which is described by the C-bracket has a doubled structure. This structure is very important to find the origin of strong constraints.

In Chapter 4, we describe the framework of the generalized geometry and the doubled geometry, which are related to T-duality. The feature of the generalized geometry is that it introduces a generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ that is a direct sum of the tangent bundle TM and the cotangent bundle T^*M. Since the symmetry group of this geometry is especially compatible with supergravity. The doubled geometry, the geometry of DFT, is also described. This geometry is fundamentally different from the generalized geometry because it doubles the degrees of freedom of the underlying base space (manifold) itself. Doubled geometry is related to the para-Hermitian geometry and the Born geometry. Then, we present the para-Hermitian manifold $\mathcal{M}$, which will be necessary for later discussion. The relationship between the generalized geometry and the para-Hermitian geometry is also discussed.

In Chapter 5, we investigate the origin of the strong constraint, which is the consistency condition of DFT. For this purpose, we use the algebroid structures introduced in the previous chapters. The Vaisman algebroid discussed in Chapter 3 is reproduced on the para-Hermitian geometry introduced in Chapter 4. First, the exterior derivatives which are necessary to give the Lie algebroids on a para-Hermitian geometry. Then, we discuss the algebraic origin of the DFT constraints by considering the double of the Lie algebroids. We also implement the other algebroid structures discussed in Chapter 3 in the same way.

In Chapter 6, we summarize this thesis and discuss the future outlook.

Appendix A contains all the calculations used to prove the doubled structures for various algebroids, including the Vaisman algebroid, which is the subject of this thesis.

2 Double Field Theory

A field theory defined on a $2D$-dimensional doubled space which is the pair of the usual momentum coordinate x combined with the winding coordinate $\tilde{x}$ is called a Double Field Theory (DFT). The field theory defined on a $2D$-dimensional doubling space is called Double Field Theory (DFT) [63]. Historically, DFT was constructed bottom-up using SFT, but it is also a recasting of SUGRA into a T-duality covariant. The DFT includes the gauge $g_{\mu\nu}$, the Kalb-Ramond field (B field) $B_{\mu\nu}$, and the dilaton ϕ as massless fields. The gauge algebra in DFT is represented by C-bracket and is a mixture of general coordinate transformations of the metric and $U(1)$ gauge transformations of the B-field.

We will leave all the mathematical background for later chapters. In this chapter, we construct DFT (in a sort of descent), which is a physical motivation to think about T-duality and related geometry. Instead, we will add annotations as appropriate to show where it is related to the later chapters of this paper, and this chapter should be used as a guide when reading this paper. First, we introduce the doubled spacetime, and also introduce the fundamental field of the type II DFT. This basic field is used to give the type II DFT action. We also discuss the properties of the C-bracket governing the generalized Lie deirivative. We discuss the constraints that hold for the DFT. We also consider the heterotic DFT set up.

2.1 Type II DFT action

DFT is defined on 2D-dimensional spacetime. Moreover, it is not just an ordinary spacetime with even dimensions, but doubled spacetime. The doubled spacetime coordinate system X^M consists of the usual momentum coordinate x^μ, which is Fourier conjugate to the KK-mode of the string, and The doubled spacetime coordinate system X^M is realized by equivalently combining the string's winding number and Fourier conjugate winding coordinate $\tilde{x}_\mu$, as follows.

$$X^M = \begin{pmatrix} \tilde{x}_\mu \\ x^\mu \end{pmatrix}, \qquad (M = 1, \dots, 2D;\ \mu = 1, \dots, D). \tag{2.1}$$

Taking T-dual corresponds to interchanging x^μ and $\tilde{x}_\mu$. Naturally, every object on doubled spacetime is a "doubled quantity". For example, for any vector on a doubled spacetime (let us call it a doubled vector) V^M is written by a summation of the part v^μ that depends on

x^μ and the part $\tilde{v}_\mu$ that depends on $\tilde{x}_\mu$, as follows.

$$V^M = v^\mu \partial_\mu + \tilde{v}_\mu \tilde{\partial}^\mu. \tag{2.2}$$

However, once the basis of doubled spacetime is written as follows.

$$\partial_M = \frac{\partial}{\partial x^M} = \begin{pmatrix} \tilde{\partial}^\mu \\ \partial_\mu \end{pmatrix}, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \ \tilde{\partial}^\mu = \frac{\partial}{\partial \tilde{x}_\mu}. \tag{2.3}$$

We also introduce the following $O(D,D)$ invariant metric for doubled spacetime.

$$\eta^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

$O(D,D)$ metric is an neutral metric. Raising or lowering the index of any doubled tensor is done by this $O(D,D)$ metric. A para-Hermitian manifold can be naturally realized by adopting such a doubled spacetime setup, which will be discussed in Chapter 4. In this chapter, it is sufficient to note that the coordinates derived from the winding mode $\tilde{x}$ and the coordinates derived from the KK-momentum $x$physically coexist. There is no problem if we only keep in mind that winding mode-derived coordinates and KK-momentum-derived coordinates coexist physically.

Then, under this basis, we introduce a fundamental dynamical field on the doubled spacetime that appears in the DFT. There are two fundamental dynamical fields in the DFT: the generalised scalar (DFT dliaton) $d(X)$ and the generalised metric $\mathcal{H}_{MN}(X)$. The generalized metric has the $O(D,D)$ constraint as follows.

$$\mathcal{H}^{MN} = \eta^{MK}\eta^{NL}\mathcal{H}_{KL}. \tag{2.5}$$

$\mathcal{H}_{MN}$ is parametrized in the coset space $O(D,D)/(O(D)\times O(D))$ as follows [87].

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}B_{\rho\nu} \\ B_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \end{pmatrix}. \tag{2.6}$$

The DFT dilaton is parametrized as

$$e^{-2d} = \sqrt{-g}e^{-2\phi}. \tag{2.7}$$

Note that g is the determinant of the metric.

The $O(D,D)$-invariant DFT action is given by the generalized Ricci scalar $\mathcal{R}$ as follows [87].

$$S_{\mathrm{DFT}} = \int \mathrm{d}^{2D}x \, e^{-2d}\mathcal{R}(\mathcal{H},d), \tag{2.8}$$

$$\begin{aligned} \mathcal{R} = {} & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ & + 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\,\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d. \end{aligned} \tag{2.9}$$

Here, $\mathrm{d}^{2D}x = \mathrm{d}^D x \mathrm{d}^D\tilde{x}$. The $\mathcal{R}$ is called generalized Ricci scalar.

2.2 Generalized Lie deivative and gauge symmetry

We introduce the generalized Lie deivative $\hat{\mathcal{L}}$. For any doubled scalar f and doubled vector V^M, it acts as follows

$$\hat{\mathcal{L}}_\Xi f = \Xi^M \partial_M f, \tag{2.10}$$
$$\hat{\mathcal{L}}_\Xi V^M = \Xi^N \partial_N V^M - (\partial^M \Xi_K - \partial_K \Xi^M) V^K. \tag{2.11}$$

where the doubled gauge parameter is Ξ^M. Also, generalized Lie deivative acts on the scalar density scalar density $\mathfrak{f}$ and vector density $\mathfrak{v}$ of the weight ω [88].

$$\hat{\mathcal{L}}_\Xi \mathfrak{f} = \Xi^M \partial_M \mathfrak{f} + \omega \mathfrak{f} \Xi^M \partial_M, \tag{2.12}$$
$$\hat{\mathcal{L}}_\Xi \mathfrak{v}^M = \Xi^N \partial_N \mathfrak{v}^M - (\partial^M \Xi_K - \partial_K \Xi^M) \mathfrak{v}^K + \omega \mathfrak{v}^K \partial_M \Xi^M. \tag{2.13}$$

Therefore, in general, $\hat{\mathcal{L}}$ for the n-tensor $T_{M_1 M_2 \cdots M_n}$ of the weight ω defined as

$$\begin{aligned}\hat{\mathcal{L}}_\Xi T^{M_1 M_2 \cdots M_n} &= \Xi^N \partial_N T^{M_1 M_2 \cdots M_n} + \omega T^{M_1 M_2 \cdots M_n} \partial_M \Xi^M \\ &\quad + \sum_{i=1}^{n} (\partial^{M_i} \Xi_K - \partial_K \Xi^{M_i}) T^{M_1 M_2 \cdots M_K \cdots M_n}.\end{aligned} \tag{2.14}$$

We discuss some more properties of $\hat{\mathcal{L}}$. $\hat{\mathcal{L}}$ is linear to the parameter Ξ. That is, when $\hat{\mathcal{L}}$ acts on the arbitary tensor product $T_1 T_2$, it becomes $\hat{\mathcal{L}}_\Xi (T_1 T_2) = (\hat{\mathcal{L}}_\Xi T_1) T_2 + T_1 (\hat{\mathcal{L}}_\Xi T_2)$. There the distributive law is satisfied. また, $\hat{\mathcal{L}}$ acts on $O(D,D)$ metric as

$$\hat{\mathcal{L}}_\Xi \eta^{MN} = \hat{\mathcal{L}}_\Xi \eta_{MN} = 0. \tag{2.15}$$

If $\hat{\mathcal{L}}_\Xi$ is a Lie derivative on doubled spacetime, the gauge parameter Ξ is a Killing vector for the $O(D,D)$ metric. Unlike the usual gauge transformation, the generalised gauge trans formation for doubled 1-forms can be given by a foot change up or down by η_{MN}.

$$\begin{aligned}\hat{\mathcal{L}}_\Xi V_M &= \hat{\mathcal{L}}_\Xi \eta_{MN} V^N \\ &= \Xi^N \partial_N V_M - (\partial_M \Xi^K - \partial^K \Xi_M) V_K.\end{aligned} \tag{2.16}$$

Using the weight of $\mathcal{H}$ is 0 and the weight of $e^{-2d} = 1$, generalized Lie derivative $\hat{\mathcal{L}}$ of generalized metric $\mathcal{H}$ and generalized dilaton d as

$$\hat{\mathcal{L}}_\Xi \mathcal{H}_{MN} = \Xi^K \partial_K \mathcal{H}_{MN} + (\partial_M \Xi^K - \partial^K \Xi_M) \mathcal{H}_{KN} + (\partial_N \Xi^K - \partial^K \Xi_N) \mathcal{H}_{MK}, \tag{2.17}$$
$$\hat{\mathcal{L}}_\Xi e^{-2d} = \Xi^K \partial_K e^{-2d} + e^{-2d} \Xi^K \partial_K. \tag{2.18}$$

Recalling the general relativity, the Lie derivative of a vector in normal spacetime can be expressed using a Lie bracket (commutator). As an analogy, we define a new bracket on the doubled space from the generalized Lie derivative as follows. This bracket is called D-bracket.

$$[\Xi_1, \Xi_2]^M_{\mathsf{D}} = \hat{\mathcal{L}}_{\Xi_1} \Xi_2^M. \tag{2.19}$$

D-bracket is a bilinear map for Ξ_1, Ξ_2, but it is a symmetric bracket. This bracket is shifted as follows when compared with the mere commutator $[\Xi_1, \Xi_2] = \Xi_1^N \partial_N \Xi_2^M - \Xi_2^N \partial_N \Xi_1^M$ (Intuitively, the commutator seems to be a "doubled Lie bracket" with doubled vectors).

$$\begin{aligned}[\Xi_1, \Xi_2]_{\mathsf{D}}^M &= [\Xi_1, \Xi_2] + \partial^M \Xi_1^N \Xi_{2N} \\ &= [\Xi_1, \Xi_2] + \eta^{MK} \eta^{NL} \partial_K \Xi_1^N \Xi_2^L. \end{aligned} \tag{2.20}$$

Since the ordinary Lie bracket is anti-symmetic, it can be seen that the D-bracket is essentially different from the Lie bracket.

We further investigate the properties of D-brackets. In general, if a symmetric bracket satisfies the Leibniz identity, it becomes a Leibnitz algebra [89,90]. We check Leibniz identity for the D-bracket. First, we calculated following terms.

$$\mathfrak{L}_{\mathsf{D}}(\Xi_1, \Xi_2, \Xi_3) = [\Xi_1, [\Xi_2, \Xi_3]_{\mathsf{D}}]_{\mathsf{D}} - \left([[\Xi_1, \Xi_2]_{\mathsf{D}}, \Xi_3]_{\mathsf{D}} + [\Xi_2, [\Xi_1, \Xi_3]_{\mathsf{D}}]_{\mathsf{D}} \right). \tag{2.21}$$

If $\mathfrak{L}_{\mathsf{D}} = 0$, the D-bracket satisfies Leibniz identity. However, There are non-zero terms remain.

$$\mathfrak{L}_{\mathsf{D}}(\Xi_1, \Xi_2, \Xi_3) = -\eta_{KL} \left(\Xi_3^K \partial^N \Xi_2^L \partial_N \Xi_1^M - \Xi_3^K \partial^N \Xi_1^L \partial_N \Xi_2^M + \Xi_2^K \partial^N \Xi_1^L \partial_N \Xi_3^M \right). \tag{2.22}$$

Therefore, D-bracket does not govern the Leibniz algebra. D-bracket is suggested to have a more fundamentally different structure.

Similarly, as an analogy for the general relativity, we expand the commutator of the generalized Lie derivative itself $[\hat{\mathcal{L}}_{\Xi_1}, \hat{\mathcal{L}}_{\Xi_2}]V^M$. Recall the exchange relation in Lie bracket for ordinary vectors, this seems to be the generalized Lie derivative of the commutator $\hat{L}_{[}\Xi_1, \Xi_2]V^M$, but in fact we obtain the following result.

$$[\hat{\mathcal{L}}_{\Xi_1}, \hat{\mathcal{L}}_{\Xi_2}]V^M = \hat{\mathcal{L}}_{[\Xi_1, \Xi_2]_{\mathsf{C}}} V^M + T^M(\Xi_1, \Xi_2, V). \tag{2.23}$$

Here, $[\Xi_1, \Xi_2]_{\mathsf{C}}$ is the following bracket. This is called C-bracket. C-bracket is an anti-symmetric bracket, unlike C-bracket.

$$[\Xi_1, \Xi_2]_{\mathsf{C}}^M = \Xi_1^K \partial_K \Xi_2^M - \Xi_2^K \partial_K \Xi_1^M - \frac{1}{2} \eta^{MN} \eta_{KL} \left(\Xi_1^K \partial_N \Xi_2^L - \Xi_2^K \partial_N \Xi_1^L \right) \tag{2.24}$$

$T^M(\Xi_1, \Xi_2, V)$ is following quantity.

$$T^M(\Xi_1, \Xi_2, V) = \frac{1}{2} \eta_{KL} \left(\Xi_1^K \partial^P \Xi_2^L - \Xi_2^K \partial^P \Xi_1^L \right) \partial_P V^M - \left(\partial^P \Xi_1^M \partial_P \Xi_2^K - \partial^P \Xi_2^M \partial_P \Xi_1^K \right) V_K. \tag{2.25}$$

If we examine the properties of the generalised Lie derivative as an analogy to the ordinary Lie derivative, we find two apparently different brackets, D-bracket and C-bracket. In fact, C-bracket is an anti-symmetric reworking of D-bracket. It is clear from the expressions (2.19) and (2.23)that they are related as follows:

$$\begin{aligned}[\Xi_1, \Xi_2]_{\mathsf{C}}^M &= \frac{1}{2} \left([\Xi_1, \Xi_2]_{\mathsf{D}}^M - [\Xi_2, \Xi_1]_{\mathsf{D}}^M \right) \\ &= [\Xi_1, \Xi_2]_{\mathsf{D}}^M - \frac{1}{2} \eta^{MN} \eta_{KL} \partial_N \left(\Xi_1^K \Xi_2^L \right) \end{aligned} \tag{2.26}$$

Using this, we calculate the Jacobiator of the C-bracket

$$\begin{aligned} J_{\mathsf{C}}(\Xi_1,\Xi_2,\Xi_3) &= [[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{C}} + [[\Xi_2,\Xi_3]_{\mathsf{C}},\Xi_1]_{\mathsf{C}} + [[\Xi_3,\Xi_1]_{\mathsf{C}},\Xi_2]_{\mathsf{C}} \\ &= [[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{C}} + \text{c.p.}. \end{aligned} \tag{2.27}$$

Here, c.p. means the cyclic parmutation. From (2.26), we obtain

$$\begin{aligned} [[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{C}} &= \frac{1}{2}([[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{D}} - [\Xi_3,[\Xi_1,\Xi_2]_{\mathsf{C}}]_{\mathsf{D}}) \\ &= \frac{1}{4}([[\Xi_1,\Xi_2]_{\mathsf{D}},\Xi_3]_{\mathsf{D}} - [[\Xi_2,\Xi_1]_{\mathsf{D}},\Xi_3]_{\mathsf{D}} - [\Xi_3,[\Xi_1,\Xi_2]_{\mathsf{D}}]_{\mathsf{D}} + [\Xi_3,[\Xi_2,\Xi_1]_{\mathsf{D}}]_{\mathsf{D}}). \end{aligned} \tag{2.28}$$

We use the D-bracket の Leibniz rule, and we obtain

$$\begin{aligned} [[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{C}} = \frac{1}{4}(&[\Xi_1,[\Xi_2,\Xi_3]_{\mathsf{D}}]_{\mathsf{D}} - [\Xi_2,[\Xi_1,\Xi_3]_{\mathsf{D}}]_{\mathsf{D}} - \mathfrak{L}_{\mathsf{D}}(\Xi_1,\Xi_2,\Xi_3) \\ &- [\Xi_2,[\Xi_1,\Xi_3]_{\mathsf{D}}]_{\mathsf{D}} + [\Xi_1,[\Xi_2,\Xi_3]_{\mathsf{D}}]_{\mathsf{D}} + \mathfrak{L}_{\mathsf{D}}(\Xi_2,\Xi_1,\Xi_3) \\ &- [\Xi_3,[\Xi_1,\Xi_2]_{\mathsf{D}}]_{\mathsf{D}} + [\Xi_3,[\Xi_2,\Xi_1]_{\mathsf{D}}]_{\mathsf{D}}). \end{aligned} \tag{2.29}$$

We can rewrite the cyclic term, so the Jacobiator becomes

$$\begin{aligned} J_{\mathsf{C}}(\Xi_1,\Xi_2,\Xi_3) &= \frac{1}{4}\left([\Xi_1,[\Xi_2,\Xi_3]_{\mathsf{D}}]_{\mathsf{D}} - [\Xi_2,[\Xi_1,\Xi_3]_{\mathsf{D}}]_{\mathsf{D}} - \mathfrak{L}_{\mathsf{D}}(\Xi_1,\Xi_2,\Xi_3) + \mathfrak{L}_{\mathsf{D}}(\Xi_2,\Xi_1,\Xi_3) + \text{c.p.}\right) \\ &= \frac{1}{4}([[\Xi_1,\Xi_2]_{\mathsf{D}},\Xi_3]_{\mathsf{D}} - \mathfrak{L}_{\mathsf{D}}(\Xi_2,\Xi_1,\Xi_3) + \text{c.p.}). \end{aligned} \tag{2.30}$$

Here, from (2.26), we obtain

$$\begin{aligned} [[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{C}} &= [[\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3]_{\mathsf{D}} - \partial^{\bullet}([\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3) \\ &= [[\Xi_1,\Xi_2]_{\mathsf{D}},\Xi_3]_{\mathsf{D}} - [\partial^{\bullet}(\Xi_1,\Xi_2),\Xi_3]_{\mathsf{D}} - \partial^{\bullet}([\Xi_1,\Xi_2]_{\mathsf{C}},\Xi_3). \end{aligned} \tag{2.31}$$

Here, $\partial^{\bullet}$ means $\eta^{MN}\partial_N$ and $(\Xi_1,\Xi_2) = (\eta_{MN}\Xi_1^M\Xi_2^N)/2$. The second term on the right-hand side of (2.31) is calculated as

$$\begin{aligned} [\partial^{\bullet}(\Xi_1,\Xi_2)_{+},\Xi_3]_{\mathsf{D}}^M &= \frac{1}{2}\Big(\partial^N(\Xi_1^K\Xi_{2,K})\partial_N\Xi_3^M + (\partial^M\partial_N(\Xi_1^K\Xi_{2,K}) - \partial_N\partial^M(\Xi_1^K\Xi_{2,K}))\Xi_3^N\Big) \\ &= \frac{1}{2}\partial^N(\Xi_1^K\Xi_{2,K})\partial_N\Xi_3^M. \end{aligned} \tag{2.32}$$

Finally, Using (2.31), we obtain the Jacobiator as

$$J_{\mathsf{C}}^M(\Xi_1,\Xi_2,\Xi_3) = \partial^M N_{\mathsf{C}}(\Xi_1,\Xi_2,\Xi_3) + \mathrm{SC}_{\mathsf{C}}(\Xi_1,\Xi_2,\Xi_3) \tag{2.33}$$

where

$$N_{\mathsf{C}}(\Xi_1,\Xi_2,\Xi_3) = \frac{1}{3}\left(([\Xi_1,\Xi_2]_{\mathsf{C}}\,,\Xi_3) + \text{c.p.}\right) \tag{2.34}$$

$$\mathrm{SC}_{\mathsf{C}}(\Xi_1,\Xi_2,\Xi_3) = \frac{1}{3}\left([\partial^{\bullet}(\Xi_1,\Xi_2),\Xi_3]_{\mathsf{D}} - \mathfrak{L}_{\mathsf{D}}(\Xi_1,\Xi_2,\Xi_3) + \text{c.p.}\right). \tag{2.35}$$

In generally, the Jacobiator of the C-bracket (2.33) is non-zero. It is clear that the C-bracket governs not Lie algebra. There is a different gauge structure.

2.3 Section constraints

In the previous section, we introduce the generalised Lie derivative, D-bracket and C-bracket, and investigate the property of these bracket. Recalling here the exchange relation of the generalised Lie derivative (2.23), the following conditions are required to close the gauge algebra described by the C-bracket,

$$\eta_{MN}\partial^M\Psi_1\partial^N\Psi_2 = 0, \tag{2.36}$$

for any field or parameter Ψ. The condition for the Jacobiator of the C-bracket is written by total deivative is also (2.36).

In addition, the condition (2.36) is also necessary to state the gauge invariance of the DFT action. The generalized deffeomorphism of generalized Ricci scalar $\mathcal{R}$ is calclated as [83]

$$\delta_\Xi\mathcal{R} = \hat{\mathcal{L}}_\Xi\mathcal{R} = \Xi^M\partial_M\mathcal{R} + G(\Xi,\mathcal{H},d) \tag{2.37}$$

where

$$\begin{aligned}G(\Xi,\mathcal{H},d) = &-\partial^P\partial_N\Xi_M\partial_P\mathcal{H}^{MN} - 2\partial^P\Xi_M\partial_P\partial_N\mathcal{H}^{MN} + 4\partial_Pd\partial_M\partial^P\Xi_N\mathcal{H}^{MN}\\ &+ 4\partial_Pd\partial^P\Xi_N\partial_M\mathcal{H}^{MN} + 4\partial_Nd\partial^P\Xi_M\partial_P\mathcal{H}^{MN}\\ &+ \frac{1}{4}\mathcal{H}^{MN}\partial^P\Xi_M\partial_P\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \mathcal{H}^{MN}\partial^P\Xi_M\partial_P\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\\ &+ 8\mathcal{H}^{MN}\partial^P\Xi_M\partial_P\partial_Nd - 8\mathcal{H}^{MN}\partial_Md\partial^P\Xi_N\partial_Pd\\ &- 2\partial_M\left(\partial^P\partial_P\Xi_N\mathcal{H}^N\right) + 4\partial^P\partial_P\Xi_M\partial_Nd\mathcal{H}^{MN}.\end{aligned} \tag{2.38}$$

Using this results, the generalised deffeomorphism of the type II DFT action is calculated as

$$\begin{aligned}\delta_\Xi S_{\mathrm{DFT}} &= \int \mathrm{d}^{2D}X\delta_\Xi\left(e^{-2d}\mathcal{R}\right) = \int \mathrm{d}^{2D}X\left(\left(\delta_\Xi e^{-2d}\right)\mathcal{R} + e^{-2d}\delta_\Xi\mathcal{R}\right)\\ &= \int \mathrm{d}^{2D}X\left(\Xi^K\left(\partial_Ke^{-2d}\right)\mathcal{R} + e^{-2d}\left(\partial_K\Xi^K\right)\mathcal{R} + e^{-2d}\Xi^K\partial_K\mathcal{R} + e^{-2d}G(\Xi,\mathcal{H},d)\right)\\ &= \int \mathrm{d}^{2D}X\left[\partial_K\left(e^{-2d}\Xi^K\mathcal{R}\right) + e^{-2d}G(\Xi,\mathcal{H},d)\right].\end{aligned} \tag{2.39}$$

Here we assume that the integral of the total derivative terms become zero. We focus on the each terms in $G(\Xi,\mathcal{H},d)$ (2.38), $\delta_\Xi S_{\mathrm{DFT}} = 0$ only by request the condition (2.36). In this Section we further consider the condition (2.36).

Historically, DFT is a theory with consistecy condition. The first version of DFT was built using arguments in closed SFT [64]. There is the level-matching condition (LMC) $p_\mu\omega^\mu = 0$. p^μ, ω^μ are the quantised momentum and winding number of the string. Implementing this in doubled spacetime, the LMC becomes

$$\partial_\mu\tilde{\partial}^\mu\Psi(x^\mu,\tilde{x}_\mu) = 0 \tag{2.40}$$

$\bar{\mathtt{x}}_\mu$	$\mathtt{x}^\mu$
$\tilde{x}_1$	x^1
x^2	$\tilde{x}_2$
$\tilde{x}_3$	x^3
$\tilde{x}_4$	x^4
$\vdots$	$\vdots$
x^D	$\tilde{x}_D$

Table 2.1: An example of the relation between $(\bar{\mathtt{x}}_\mu, \mathtt{x}^\mu)$ and $(\tilde{x}_\mu, x^\mu)$.

where $\Psi(x^\mu, \tilde{x}_\mu)$ is arbitary massless fields. This constraint is called *weak constraint*. Any field or gauge parameter appearing in the DFT must satisfy this constraint. However, in general, arbitrary product of fields do not satisfy this constraint. This problem can be solved by introducing projection operators on product of fields.. Indeed, the construction of DFTs based on weak constaint is discussed, for example, in [91]. However, the most common way is to introduce the following other constraint, which is effective for arbitrary products of fields.

$$\partial_\mu \tilde{\partial}^\mu (\Psi_1 \Psi_2) = 0. \tag{2.41}$$

This constraint ie called *strong constraint*. This can be rewritten to $O(D, D)$ covariant form, it becomes (2.36). The strong constraint has no physical origin and is only the closure condition of the C-bracket, as described at the beginning of this Section. The relaxation of strong constraints is discussed, for example, in [83, 88]. In this paper, Sec. 5.6 are also relevant.

Strictly speaking, DFT should be interpreted as a theory on $\mathbb{R}^{D-n} \times T^{2n}$ (T^{2n} is a doubled torus in $2n$-dimensions). Given the connection from string theory, only $O(n, n, \mathbb{Z})$ group acts on the doubled torus as a T-duality group, which is expected to reproduce the T-duality in string theory. However, DFT with strong constraints imposed, $O(n, n, \mathbb{Z})$ can be extended to $O(D, D, \mathbb{R})$ acting on the form $\mathbb{R}^{2D}$. This is because, a set of fields satisfying strong constraints always have an $O(D, D)$ frame $(\bar{\mathtt{x}}_\mu, \mathtt{x}^\mu)$ which the fields depends only on $\mathtt{x}^\mu$ [92]. In this situation, $(\bar{\mathtt{x}}_\mu, \mathtt{x}^\mu)$ simply means that the inner product is formed by $O(D, D)$ metric η (Different characters $\bar{\mathtt{x}}, \mathtt{x}$ are used to distinguish it from $\tilde{x}, x$). In other words, DFT imposed the strong constraint does not actually mean that the theory is truly "doubled".

We refer to the weak and strong constraints collectively as the *physically section condition*. As mentioned above, the simplest way to solve this condition is to make the field depend only on $\mathtt{x}^\mu$) out of $(\bar{\mathtt{x}}_\mu, \mathtt{x}^\mu)$. Although the display is inevitably confusing, this only selects a D-dimensional subspace from a $2D$-dimensional space. Therefore, $\bar{\mathtt{x}}_\mu$ and $\mathtt{x}^\mu$ have no particular physical meaning. In general, $\mathtt{x}^\mu$ contains x^μ and $\tilde{x}_\mu$. For pairs constituting an $O(D, D)$ inner product with η (e.g. x^1 and $\tilde{x}_1$), only one of them is included (Figure 2.1).

Thus, by imposing the section condition, a maximally isotropic subspace of half dimension can be extracted from a $2D$-dimentional doubled space. In this case, the coordinates selected as the subspace depend on how the T-duality frame is taken. Therefore, the $O(D,D)$ transformation can change the coordinates selected as the subspace. This idea of identifying spacetime as a subspace determined in terms of T-duality is described in [54]. This corresponds to selecting a leaf from the foliation in para-Hermitian geometry (see Chapter 4). It also corresponds, in physical terms, to selecting the coordinates corresponding to the physical spacetime when we consider the supergravity from the DFT. Once, we consider the simplest example. We choose the D-dimentional subspace subspace as $\mathtt{x}^\mu = x^\mu$. Then, since all fields do not depend on $\tilde{x}_\mu$, we have

$$\frac{\partial}{\partial \tilde{x}_\mu}\Psi(x) = \tilde{\partial}^\mu \Psi(x) = 0. \tag{2.42}$$

We rewrite the generalized Ricci scalar $\mathcal{R}$ according to this condition and find that DFT action is reduced to the bosonic part of type II super gravity.

$$S_{\mathrm{DFT}} = \int \mathrm{d}X e^{-2d}\mathcal{R} \overset{\tilde{\partial}\Psi=0}{\longrightarrow} S_{\mathrm{sugra}} = \int \mathrm{d}x\sqrt{-g}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right). \tag{2.43}$$

where R is the usual Ricci scalar that appears in the general rerativity. Also, $H = dB$ is the field strength of the Kalb-Ramond field $B_{\mu\nu}$. In this sense, (type II) DFT is an extension of type II supergravity in the form of explicit T-duality.

The C-bracket is also deformated by strong constraint. To make this explicit, we first decompose the gauge parameters Ξ_1, Ξ_2 by their components as follows.

$$\Xi_1^M = \begin{pmatrix} \alpha_\mu \\ A^\mu \end{pmatrix}, \qquad \Xi_2^M = \begin{pmatrix} \beta_\mu \\ B^\mu \end{pmatrix}. \tag{2.44}$$

Here, α, β are parameters on $\bar{\mathtt{x}}_\mu$ and A, B are parameters on $\mathtt{x}_\mu$. With this component representation, the C-bracket can be rewritten as follows.

$$\begin{aligned}
[\Xi_1, \Xi_2]_{\mathsf{C}}^M &= \Xi_1^K \partial_K \Xi_2^M - \Xi_2^K \partial_K \Xi_1^M - \frac{1}{2}\eta_{KL}(\Xi_1^K \partial^M \Xi_2^L - \Xi_2^K \partial^M \Xi_1^L) \\
&= \alpha_\nu \tilde{\partial}^\nu \Xi_2^M + A^\nu \partial_\nu \Xi_2^M - \beta_\nu \tilde{\partial}^\nu \Xi_1^M - B^\nu \partial_\nu \Xi_1^M \\
&\quad - \frac{1}{2}(\alpha_\nu \partial^M B^\nu + A^\nu \partial^M \beta_\nu - \beta_\nu \partial^M A^\nu - B^\nu \partial^M \alpha_\nu).
\end{aligned} \tag{2.45}$$

Using the doubled basis $\partial^M = (\partial_\mu, \tilde{\partial}^\mu)$, C-bracket is expanded as

$$\begin{aligned}
[\Xi_1, \Xi_2]_{\mathsf{C}} = [\Xi_1, \Xi_2]_{\mathsf{C}}^M \eta_{MN}\partial^N &= \alpha_\nu \tilde{\partial}^\nu \beta_\mu \tilde{\partial}^\mu + A^\nu \partial_\nu \beta_\mu \tilde{\partial}^\mu - \beta_\nu \tilde{\partial}^\nu \alpha_\mu \tilde{\partial}^\mu - B^\nu \partial_\nu \alpha_\mu \tilde{\partial}^\mu \\
&\quad + \alpha_\nu \tilde{\partial}^\nu B^\mu \partial_\mu + A^\nu \partial_\nu B^\mu \partial_\mu - \beta_\nu \tilde{\partial}^\nu A^\mu \partial_\mu - B^\nu \partial_\nu A^\mu \partial_\mu \\
&\quad - \frac{1}{2}(\alpha_\nu \tilde{\partial}^\mu B^\nu + A^\nu \tilde{\partial}^\mu \beta_\nu - \beta_\nu \tilde{\partial}^\mu A^\nu - B^\nu \tilde{\partial}^\mu \alpha_\nu)\partial_\mu \\
&\quad - \frac{1}{2}(\alpha_\nu \partial_\mu B^\nu + A^\nu \partial_\mu \beta_\nu - \beta_\nu \partial_\mu A^\nu - B^\nu \partial_\mu \alpha_\nu)\tilde{\partial}^\mu.
\end{aligned} \tag{2.46}$$

We shall now recapitulate each term e.g. in the commutator as follows,

$$
\begin{aligned}
[A,B]_L &= [A,B]_L^\mu \partial_\mu = (A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu)\partial_\mu, \\
[\alpha,\beta]_{\tilde{L}} &= \left([\alpha,\beta]_{\tilde{L}}\right)_\mu \tilde{\partial}^\mu = (\alpha_\nu \tilde{\partial}^\nu \beta_\mu - \beta_\nu \tilde{\partial}^\nu \alpha_\mu)\tilde{\partial}^\mu, \\
\mathrm{d}\iota_A\beta &= \mathrm{d}(A^\nu\beta_\nu) = \partial_\mu(A^\nu\beta_\nu)\tilde{\partial}^\mu = (\beta_\nu\partial_\mu A^\nu + A^\nu\partial_\mu\beta_\nu)\tilde{\partial}^\mu, \\
\tilde{\mathrm{d}}\iota_A\beta &= \tilde{\mathrm{d}}(A^\nu\beta_\nu) = \tilde{\partial}^\mu(A^\nu\beta_\nu)\partial_\mu = (\beta_\nu\tilde{\partial}^\mu A^\nu + A^\nu\tilde{\partial}^\mu\beta_\nu)\partial_\mu, \\
\tilde{\mathcal{L}}_\alpha B &= (\alpha_\nu\tilde{\partial}^\nu B^\mu + B^\nu\tilde{\partial}^\mu\alpha_\nu)\partial_\mu, \\
\mathcal{L}_A\beta &= (A^\nu\partial_\nu\beta_\mu + \beta_\nu\partial_\mu A^\nu)\tilde{\partial}^\mu,
\end{aligned}
\tag{2.47}
$$

where $\mathcal{L}$ is the Lie derivative with respect to an ordinary vector field. On the other hand, $\tilde{\mathcal{L}}$ is the Lie derivative with respect to a vector field in winding coordinates. Similarly,$[\cdot,\cdot]_L$is an ordinary Lie bracket, but Similarly, $[\cdot,\cdot]_{\tilde{L}}$is a Lie bracket with respect to the vector field in winding coordinates. Using these, we can rewrite the C-bracket as

$$
\begin{aligned}
[\Xi_1,\Xi_2]_{\mathsf{C}}^M\partial_M &= ([\alpha,\beta]_{\tilde{L}})_\mu\tilde{\partial}^\mu + A^\nu\partial_\nu\beta_\mu\tilde{\partial}^\mu - B^\nu\partial_\nu\alpha_\mu\tilde{\partial}^\mu + [A,B]_L^\mu\partial_\mu + \alpha_\nu\tilde{\partial}^\nu B^\mu\partial_\mu - \beta_\nu\tilde{\partial}^\nu A^\mu\partial_\mu \\
&\quad - \frac{1}{2}(2A^\nu\tilde{\partial}^\mu\beta_\nu - (\tilde{\mathrm{d}}\iota_A\beta)^\mu + (\tilde{\mathrm{d}}\iota_B\alpha)^\mu - 2B^\nu\tilde{\partial}^\mu\alpha_\nu)\partial_\mu \\
&\quad - \frac{1}{2}((\mathrm{d}\iota_A\beta)_\mu - 2\beta_\nu\partial_\mu A^\nu - (\mathrm{d}\iota_B\alpha)_\mu + 2\alpha_\nu\partial_\mu B^\nu)\tilde{\partial}^\mu \\
&= \left([A,B]_L^\mu + \tilde{\mathcal{L}}_\alpha B^\mu - \tilde{\mathcal{L}}_\beta A^\mu + \frac{1}{2}(\tilde{\mathrm{d}}(\iota_A\beta - \iota_B\alpha))^\mu\right)\partial_\mu \\
&\quad + \left(([\alpha,\beta]_{\tilde{L}})_\mu + \mathcal{L}_A\beta_\mu - \mathcal{L}_B\alpha_\mu - \frac{1}{2}(\mathrm{d}(\iota_A\beta - \iota_B\alpha))_\mu\right)\tilde{\partial}^\mu.
\end{aligned}
\tag{2.48}
$$

This form can eventually be rewritten as follows [93].

$$
\begin{aligned}
[\Xi_1,\Xi_2]_{\mathsf{C}} = [A+\alpha, B+\beta]_{\mathsf{C}} &= [A,B]_L + \mathcal{L}_A\beta - \mathcal{L}_B\alpha - \frac{1}{2}\mathrm{d}(\iota_A\beta - \iota_B\alpha) \\
&\quad + [\alpha,\beta]_{\tilde{L}} + \tilde{\mathcal{L}}_\alpha B - \tilde{\mathcal{L}}_\beta A + \frac{1}{2}\tilde{\mathrm{d}}(\iota_A\beta - \iota_B\alpha).
\end{aligned}
\tag{2.49}
$$

In (2.49), the first line depends on x, the second line on $\tilde{x}$. If we impose strong constraints on (2.49), only the second line is vanished and the bracket changes as follows.

$$
[\Xi_1,\Xi_2]_{\mathsf{C}} \xrightarrow{\tilde{\partial}\Psi=0} [A,B]_L + \mathcal{L}_A\beta - \mathcal{L}_B\alpha - \frac{1}{2}\mathrm{d}(\iota_A\beta - \iota_B\alpha) \tag{2.50}
$$

Similarly, the D-bracket with the strong contraint as follows.

$$
[\Xi_1,\Xi_2]_{\mathsf{D}} \xrightarrow{\tilde{\partial}\Psi=0} [A,B]_L + \mathcal{L}_A\beta - \iota_B\mathrm{d}\alpha. \tag{2.51}
$$

In fact, the structures descrived by C-bracket and D-bracket, i.e. gauge algebra in DFT, is not algebra but algebroid. The properties of algebroid will be dealt with in the next Chapter 3. The algebraic origin of the strong constraint are discussed in Chapter 5. The structure algebroid is known to appear not only in type II DFT but also in other theories focusing on string duality [94–97] In particular, in heterotic DFT, twisted algebrids with 3-forms appear, as presented in [84, 86]. In the next section, we conclude this chapter with a brief description of the structure of heterotic DFT.

2.4 Heterotic DFT

As mentioned in Chapter 1, string duality links the five string theories. Among them, the type II DFT, described in the previous section, formulated by focusing on T-duality. T-duality relates not only type II theories but also heterotic theories [32] . It is natural to consider the construction of heterotic DFT in the same way as type II DFT. A heterotic string consists of the supersymmetric right-moving part and the left-moving part describing the gauge symmetry [98]. For the theory to be consistent, the gauge group must be $E_8 \times E_8$ or $SO(32)$. The low-energy limit of heterotic string theory is the heterotic supergravity. To realise the anomaly cancellation mechanism, heterotic supergravity requires α'-(or higher order) corrections [99,100]. The α'-corrections to DFT is also naturally discussed [81], in the process, heterotic DFT (or gauged DFT) is proposed in [82,83,101]. This can be interpreted as an $O(D, D+n)$ covariant formulation of the heterotic supergravity. Here is a very brief introduction to the construction of the heterotic DFT.

Comparing the heterotic DFT with the type II DFT, the symmetry group is extended to $O(D, D+n)$. This n corresponds to the degrees of freedom of the gauge field. The "doubled spacetime" in heterotic DFT is extended to $2D+n$ dimensions $X^M = (\tilde{x}_\mu, y_\alpha, x^\mu)$. Here, $M = 1, \cdots (2D+n), \mu, \nu = 1, \cdots, D, \alpha = 1 \cdots n$. y_α direction is the extended space by the gauge group. At this point, it can be seen that the "doubled spacetime of heterotic DFT" (rather than generalized spacetime) is fundamentally different from it of type II DFT.

We introduce the $O(D, D+n)$ invariant metritc as

$$\eta_{MN} = \begin{pmatrix} 0 & 0 & \delta^\mu{}_\nu \\ 0 & \kappa^{\alpha\beta} & 0 \\ \delta_\mu{}^\nu & 0 & 0 \end{pmatrix}, \quad \eta^{MN} = \begin{pmatrix} 0 & 0 & \delta_\mu{}^\nu \\ 0 & \kappa_{\alpha\beta} & 0 \\ \delta^\mu{}_\nu & 0 & 0 \end{pmatrix} \tag{2.52}$$

where $\kappa_{\alpha\beta}, \kappa^{\alpha\beta}$ is $n \times n$ matrix and the Cartan-Killing form of $SO(32)$ and $E8 \times E8$ group. Next, we introduce the fundamental dynamical fields $\mathcal{H}(X)$ and $d(X)$. $\mathcal{H}^{MN} = \eta^{MP}\eta^{NQ}\mathcal{H}_{PQ}$ is satisfied between the generalized metric $\mathcal{H}(X)$ and the $O(D, D+n)$ metric. The generalised metric $\mathcal{H}$ is parametrized on the coset space $O(D, D+n)/(O(D) \times O(D+n))$, including the gauge field $A_\mu{}^\alpha$.

$$\mathcal{H}^{MN} = \begin{pmatrix} g_{\mu\nu} + \alpha' A^\alpha_\mu A_{\nu\alpha} + c_{\rho\mu} g^{\rho\sigma} c_{\sigma\nu} & \sqrt{\alpha'} A_{\mu\alpha} + \sqrt{\alpha'} A_{\rho\alpha} g^{\rho\sigma} c_{\sigma\mu} & -c_{\rho\mu} g^{\rho\nu} \\ \sqrt{\alpha'} A_{\nu\beta} + \sqrt{\alpha'} A_{\rho\beta} g^{\rho\sigma} c_{\sigma\nu} & \kappa_{\alpha\beta} + \alpha' A_{\rho\alpha} g^{\rho\sigma} A_{\sigma\beta} & -\sqrt{\alpha'} A_{\rho\beta} g^{\rho\nu} \\ -c_{\sigma\nu} g^{\sigma\mu} & -\sqrt{\alpha'} A_{\sigma\alpha} g^{\sigma\mu} & g^{\mu\nu} \end{pmatrix} \tag{2.53}$$

where $c_{\mu\nu} = B_{\mu\nu} + (\alpha' A_\mu{}^\alpha A_{\nu\alpha})/2$.

Heterotic DFT action is given by [82, 83]

$$S_{hDFT} = \int \mathrm{d}^{2D+n} X e^{-2d} \left(\mathcal{R}(\mathcal{H}, d) + \mathcal{R}_f(\mathcal{H}, d) \right). \tag{2.54}$$

where

$$\begin{aligned}\mathcal{R}(\mathcal{H},d) &= 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d\\ &\quad + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL},\end{aligned} \tag{2.55}$$

$$\begin{aligned}\mathcal{R}_f(\mathcal{H},d) &= -\frac{1}{2}f^M{}_{NK}\mathcal{H}^{NL}\mathcal{H}^{KP}\partial_L\mathcal{H}_{MP} - \frac{1}{12}f^M{}_{NK}f^L{}_{PQ}\mathcal{H}_{ML}\mathcal{H}^{NP}\mathcal{H}^{KQ}\\ &\quad - \frac{1}{4}f^M{}_{NK}f^N{}_{ML}\mathcal{H}^{KL} - \frac{1}{6}f^{MNK}f_{MNK}.\end{aligned} \tag{2.56}$$

Here, $\mathrm{d}^{2D+n} = \mathrm{d}^D x \mathrm{d}^D \tilde{x} \mathrm{d}^n y$ and $e^{-2d} = \sqrt{-g}e^{2\psi}$. $f^M{}_{NK}$ is the structure constant of the gauge group. $f^M{}_{NK}$ satisfies teh following conditions,

$$f^{(M}{}_{PK}\eta^{N)K} = 0, \quad f^M_{N[K}f^N{}_{LP]} = 0. \tag{2.57}$$

In particular, the second condition is the Jacobi identity for $f^M{}_{NK}$. The generalised gauge transformation $\hat{\delta}_\Xi$ for any V^M, V_M can be written as follows.

$$\hat{\delta}_\Xi V^M = \Xi^N\partial_N V^M - (\partial^M\Xi_K - \partial_K\Xi^M)V^K - \Xi^N f^M{}_{NK}V^K, \tag{2.58}$$

$$\hat{\delta}_\Xi V_M = \Xi^N\partial_N V_M - (\partial_M\Xi^K - \partial^K\Xi_M)V_K - \Xi^K f^N{}_{KM}V^N. \tag{2.59}$$

Therefore, generalized gauge transformation of $\mathcal{H}^{MN}$ and d are given by

$$\begin{aligned}\delta_\Xi\mathcal{H}^{MN} &= \Xi^P\partial_P\mathcal{H}^{MN} + \left(\partial^M\Xi_P - \partial_P\Xi^M\right)\mathcal{H}^{PN}\\ &\quad + \left(\partial^N\Xi_P - \Xi_P\Xi^N\right)\mathcal{H}^{MP} - 2\Xi^P f^{(M}{}_{PK}\mathcal{H}^{N)K},\end{aligned} \tag{2.60}$$

$$\delta_\Xi d = \Xi^M\partial_M d - \frac{1}{2}\partial_M\Xi^M. \tag{2.61}$$

Compared with type II DFT, These are deformed by $f^M{}_{NK}$. For a hetertotic DFT action (2.54) to be invariant under this generalized gauge transformation, the following conditions are required.

$$\eta_{MN}\partial^M\Psi_1\partial^N\Psi_2 = 0, \tag{2.62}$$

$$f^M{}_{NK}\partial_M\Psi_1 = 0. \tag{2.63}$$

$\Psi_i (i=1,2)$ are arbitrary fields or gauge parameters. The(2.62) is apparently same form as the strong constraint (2.36) of the type II DFT. However, η_{MN} is an $O(D, D+n)$ metric and $M, N = 1, \cdots, (2D+n)$. (2.63) is a condition on $f^M{}_{NK}$, which appeared only heterotic DFT. These conditions are the closure conditions of the gauge algebra in heterotic DFT the commutator of $\hat{\delta}_\Xi$ is calclated as

$$\begin{aligned}\left[\delta_{\Xi_1}, \delta_{\Xi_2}\right]V^M &= \delta_{[\Xi_1,\Xi_2]_f}V^M + T^M\left(\Xi^1,\Xi^2,V\right)\\ &\quad - \eta^{MR}\Xi_1^P f^Q{}_{RP}\left(\partial_Q\Xi_{2N}\right)V^N + \eta^{MR}\Xi_2^P f^Q{}_{RP}\left(\partial_Q\Xi_{1N}\right)V^N\\ &\quad + f^L{}_{NK}\left(\partial_L\Xi_1^M\right)\Xi_2^N V^K - f^L{}_{NK}\left(\partial_L\Xi_2^M\right)\Xi_1^N V^K - \Xi_2^N\Xi_1^K f^P{}_{NK}\partial_P V^M\\ &\quad + \frac{1}{2}\eta^{MR}\Xi_1^L f^N{}_{RK}\left(\partial_N\Xi_{2L}\right)V^K - \frac{1}{2}\eta^{MR}\Xi_2^L f^N{}_{RK}\left(\partial_N\Xi_{1L}\right)V^K.\end{aligned} \tag{2.64}$$

Here, $[\cdot,\cdot]_f$ is the C-bracket in heterotic DFT. It taked the following form [82, 84].

$$\begin{aligned}[\Xi_1,\Xi_2]_f &= \Xi_1^K\partial_K\Xi_2^M - \Xi_2^K\partial_K\Xi_1^M - \frac{1}{2}\eta^{MN}\eta_{KL}(\Xi_1^K\partial_K\Xi_2^L - \Xi_2^K\partial_K\Xi_1^L) + \Xi_2^N\Xi_1^K f^M{}_{NK} \\ &= [\Xi_1,\Xi_2]_{\mathsf{C}}^M + \Xi_2^N\Xi_1^K f^M{}_{NK}. \end{aligned} \tag{2.65}$$

The algebroid structure given by $[\Xi_1,\Xi_2]_f$ is discussed in Sec 5.6. Also,

$$T^M(\Xi_1,\Xi_2,V) = \frac{1}{2}\eta_{KL}(\Xi_1^K\partial^P\Xi_2^L - \Xi_2^K\partial^P)\partial_P V^M - (\partial^P\Xi_1^M\partial_P\Xi_2^K - \partial^P\Xi_2^M\partial_P\Xi_1^K)V_K. \tag{2.66}$$

First, for $T^M(\Xi_1,\Xi_2,V)$ to vanish, we need to impose (2.62). Furthermore, for the gauge algebra to close, (2.63) is also required. This additional condition can also be interpreted as a strong constraint in the direction y_α [82], but the physical origin is still unknown. If we impose the two conditions (2.62) and (2.63) for the heterotic DFT action and choice $n = 496$, then It is known that (excluding the square of the Riemannian curvature) a 10-dimensional heterotic supergravity action appears [83].

$$\begin{aligned} S_{\mathrm{hDFT}} &= \int \mathrm{d}X e^{-2d}(\mathcal{R}+\mathcal{R}_f) \overset{(2.62),(2.63)}{\longrightarrow} \\ S_{\mathrm{hsugra}} &= \int \mathrm{d}x\sqrt{-g}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{\alpha'}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}\right). \end{aligned} \tag{2.67}$$

However, R is the usual Ricci scalar that appears in the general rerativity. H is the field strength of the Kalb-Ramond field $B_{\mu\nu}$ and F is the field strength of the gauge field A_μ^α.

Now, the geometry of heterotic DFT is more obscure than that of type II. There is a doubled α' geometry in the sense that it is purely a geometry with y_α directions, but there is no global geometry such as the para-Hermitian or Born geometry for type II DFT.

3 Algebroid Structures

In this chapter, we introduces two new concepts. The first is the concept of a (classical) Drinfel'd double. Drinfel'd double was originally invented for Hopf algebras. A new Hopf algebra can be obtained by a direct sum of dual Hopf algebras, this operation is called Drinfel'd double. In a nutshell, a Hopf algebra is a parametric deformation of a Lie algebra, and a Lie algebra is obtained by taking the classical limit on the parameters. In this article, we first show that the Drinfel'd double of Lie algebra is actually performed as an example of Drinfel'd double, and that a new Lie algebra is obtained as a result.

The second new concept is the "algebroid", which was introduced in the previous chapter as a DFT gauge algebra defined by C-bracket. The most basic algebroid is the Lie algebroid. This can be regarded as a generalization of Lie algebra to vector bundles. Just as Drinfel'd double can be implemented for Lie algebra, we can consider the Drinfel'd double for Lie algebroid [102]. The structure defined by the C-bracket in DFT is the Vaisman algebroid, this is more general structure of Lie algebroid. We show that the Vaisman algebroid can be obtained by similar to Drinfel'd double on Lie algebroid [85]. This operation is referred to in the text simply as "doubled". The proof will be discussed in Sec. 3.5. (and Appendix A).

3.1 Lie algebra and Lie bialgebra

In this section, we give a brief review on the Drinfel'd double of Lie bialgebras. This is one of the important topic in this paper. Please also refer to this literature [103, 104] We first introduce the notion of Lie algebras. Lie algebra is defined as follows.

Definition 3.1.1. Let $(V, [\cdot, \cdot])$ be a Lie algebra over a field K defined by a vector space V together with a skew-symmetric bilinear bracket (the Lie bracket) $[\cdot, \cdot] : V \times V \to V$ satisfying the Jacobi identity,

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \tag{3.1}$$

We denote a Lie algebra by $\mathfrak{g}$.

Since V is a vector space, we can also define the dual Lie algebra by the dual vector space V^*. We define the dual Lie algebra$\mathfrak{g}^*$ based on the dual vector space V^* equipped with the dual Lie bracket $[\cdot, \cdot]_*$. There is a natural bilinear inner product $\langle \cdot, \cdot \rangle$ between $\mathfrak{g}$ and $\mathfrak{g}^*$ taking value in K is defined.

Next, we consider a representation ϱ of $\mathfrak{g}$. A Lie algebra element $x \in \mathfrak{g}$ acts on itself by the adjoint representation $\mathrm{ad} : x \in \mathfrak{g} \mapsto \mathrm{ad}_x \in \mathrm{End}\, \mathfrak{g}$ by $\mathrm{ad}_x(y) = [x, y]$ for $x, y \in \mathfrak{g}$. More generally, $x \in \mathfrak{g}$ acts on any tensor products $\otimes^p \mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ as

$$\begin{aligned}\varrho(x) \cdot (y_1 \otimes \cdots \otimes y_p) &= \mathrm{ad}_x^{(p)}(y_1 \otimes \cdots \otimes y_p) \\ &= \mathrm{ad}_x(y_1) \otimes y_2 \cdots \otimes y_p + y_1 \otimes \mathrm{ad}_x(y_2) \otimes \cdots \otimes y_p + \cdots \\ &\quad \cdots + y_1 \otimes \cdots \otimes \mathrm{ad}_x(y_p).\end{aligned} \tag{3.2}$$

Therefore the adjoint action satisfies the Leibniz rule. The Jacobi law (3.1) of the Lie bracket product can also be interpreted as the Leibniz law of ad.

$$\mathrm{ad}_z([x, y]) = [\mathrm{ad}_z(x), y] + [x, \mathrm{ad}_z(y)]. \tag{3.3}$$

Similarly, we consider the action of $x \in \mathfrak{g}$ on a p-order outer product algebra $\wedge^p \mathfrak{g}$. It is sufficient to consider the action on the fully antisymmetric tensor product $\otimes^p \mathfrak{g}$, which can be defined as follows.

$$\varrho(x) \cdot y_1 \wedge y_2 = [x, y_1] \wedge y_2 + y_1 \wedge [x, y_2]. \tag{3.4}$$

If we consider this in the same way as the exterior differential operator d on the cotangent bundle, we can define the operator $\mathrm{d} : \wedge^p \mathfrak{g}^* \to \wedge^{p+1} \mathfrak{g}^*$.It satisfies $\mathrm{d}^2 = 0$. In a similar procedure, by considering the representation of $\mathfrak{g}^*$ and its action, we can also introduce the dual operator $\mathrm{d}_* : \wedge^p \mathfrak{g} \to \wedge^{p+1} \mathfrak{g}$. It is satisfies $\mathrm{d}_*^2 = 0$. Using d and d^*, we can define the Lie algebra cohomology on $\mathfrak{g}$ [105].

It is worthwhile to discuss a generalization of the Lie bracket to the one in $\wedge^p \mathfrak{g}$. The skew-symmetric Schouten-Nijenhuis bracket $[\cdot, \cdot]_{\mathrm{S}} : \wedge^p \mathfrak{g} \times \wedge^q \mathfrak{g} \to \wedge^{p+q-1} \mathfrak{g}$ is defined by the following properties [106]:

(i) $[a, b]_{\mathrm{S}} = -(-)^{(p-1)(q-1)}[b, a]_{\mathrm{S}}$.

(ii) $[a, b \wedge c]_{\mathrm{S}} = [a, b]_{\mathrm{S}} \wedge c + (-)^{(p-1)q} b \wedge [a, c]_{\mathrm{S}}$.

(iii) $(-)^{(p-1)(r-1)}[a, [b, c]_{\mathrm{S}}]_{\mathrm{S}} + (-)^{(q-1)(r-1)}[b, [c, a]_{\mathrm{S}}]_{\mathrm{S}} + (-)^{(r-1)(q-1)}[c, [a, b]_{\mathrm{S}}]_{\mathrm{S}} = 0$.

(iv) The bracket of an element $\wedge^p \mathfrak{g}$ and an element in $\wedge^0 \mathfrak{g} = K$ is 0.

Here $a \in \wedge^p \mathfrak{g}$, $b \in \wedge^q \mathfrak{g}$ and $c \in \wedge^r \mathfrak{g}$. Indeed, the Schouten-Nijenhuis bracket is a unique generalization of the Lie bracket that makes $\wedge^p \mathfrak{g}$ be a Gerstenhaber algebra.

Next, we investigate the relation between $\mathfrak{g}$ and$\mathfrak{g}^*$. If a, b are element of $\mathfrak{g}$, the Lie bracket $[a, b]$ is also an element of $\mathfrak{g}$, so Lie bracket can be regarded as a bilinear map $\mu : \wedge^2 \mathfrak{g} \to \mathfrak{g}$. Since Lie bracket is antisymmetric, let $[\cdot, \cdot]$ be $\mu : \wedge^2 \mathfrak{g} \to \mathfrak{g}$. The dual Lie bracket $[\cdot, \cdot]_*$ can be written as $\mu^* : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$. We can then define the co-bracket of μ is $\delta : \mathfrak{g} \to \wedge^2 \mathfrak{g}$. The adjoint of a map μ_*, denoted as μ_*^*, is defined through the inner product $\langle \cdot, \cdot \rangle$ between $\wedge^\bullet \mathfrak{g}$ and $\wedge^\bullet \mathfrak{g}^*$ by $\langle x, \mu_*(\xi) \rangle = \langle \mu_*^*(x), \xi \rangle$ where $x \in \mathfrak{g}$ and $\xi \in \wedge^2 \mathfrak{g}^*$. Here, $\wedge^\bullet$ stands for any powers of the wedge products. Since δ is given by μ_*^*, δ can be defined by μ_*^*. Since the dual Lie bracket μ_* satisfies the Jacobi law δ satisfies the Jacobi identity. In other words, the dual Lie bracket μ_* can be defined by δ^*. If we define the Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$ and the co-bracke of $[\cdot,\cdot]$, then The dual Lie algebra $\mathfrak{g}^* = (V^*, [\cdot,\cdot]_*)$ can be induced naturally by inner product.

Definition 3.1.2. if δ satisfies the 1-cocycle condition:

$$\delta([x,y]) = \mathrm{ad}_x^{(2)}\delta(y) - \mathrm{ad}_y^{(2)}\delta(x), \quad x, y \in \mathfrak{g}, \tag{3.5}$$

then, the structure $(\mathfrak{g}, \mu, \delta)$ is called the *Lie bialgebra.*

If $(\mathfrak{g}, \mu, \delta)$ is Lie bialgebra, $(\mathfrak{g}^*, \mu^*, \delta^*)$ is also define the same Lie bialgebra. So, we write the Lie bialgebra as $(\mathfrak{g}, \mathfrak{g}^*)$.

3.2 Drinfel'd double

Next, We consider the Drinfel'd double of Lie bialgebra. For $\mathfrak{g}, \mathfrak{g}^*$ which constitute the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we can define a non-degenerate, symmetric bilinear form $(\mathfrak{g}, \mathfrak{g}^*)$ on $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ as follows,

$$(x, y) = (\xi, \eta) = 0, \qquad (x, \xi) = \langle \xi, x\rangle, \qquad x, y \in \mathfrak{g},\ \xi, \eta \in \mathfrak{g}^*. \tag{3.6}$$

We then require that there is a skew-symmetric bracket $[\cdot,\cdot]_{\mathfrak{d}}$ which is invariant under the bilinear form.

$$(y, [x,\xi]_{\mathfrak{d}}) = ([y,x]_{\mathfrak{d}}, \xi). \tag{3.7}$$

$\mathfrak{g}, \mathfrak{g}^*$ are subalgebra of $\mathfrak{d}$ respectively, this is natural definition of the bracket

$$[x,y]_{\mathfrak{d}} = [x,y], \quad [\xi,\eta]_{\mathfrak{d}} = [\xi,\eta]_*, \qquad x, y \in \mathfrak{g},\ \xi, \eta \in \mathfrak{g}^*. \tag{3.8}$$

For closs term of $\mathfrak{g}$ and $\mathfrak{g}^*$, we consider $[x,\xi]_{\mathfrak{d}}$ as follows,

$$\begin{aligned}(y, [x,\xi]_{\mathfrak{d}}) &= ([y,x]_{\mathfrak{d}}, \xi) \\ &= ([y,x], \xi) = \langle \xi, [y,x]\rangle = \langle \xi, -\mathrm{ad}_x(y)\rangle = \langle \mathrm{ad}_x^*\xi, y\rangle = (y, \mathrm{ad}_x^*\xi).\end{aligned} \tag{3.9}$$

Here, The first equality follows from the definition of the invariance. The second comes from the fact that $\mathfrak{g}$ is a subalgebra (3.8). Here we also use the co-adjoint for $\mathrm{ad}_x^* = -(\mathrm{ad}_x)^*$, Similarly, we have $(\eta, [x,\xi]_{\mathfrak{d}}) = -(\eta, \mathrm{ad}_\xi^* x)$. These facts result in the definition,

$$[x,\xi]_{\mathfrak{d}} = -\mathrm{ad}_\xi^* x + \mathrm{ad}_x^*\xi. \tag{3.10}$$

From (3.5), (3.8), and (3.10), $[\cdot,\cdot]_{\mathfrak{d}}$ is also satisfiy Jacobi identity. Thus, If the 1-cocycle condition (3.5) holds between $\mathfrak{g}$ and $\mathfrak{g}^*$ ($(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra), $(\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{d}})$ becomes an new Lie algebra. in this way, the operation of constructing a new Lie algebra $\mathfrak{d}$ is called the Drinfel'd double of Lie bialgebras. The pair of a non-degenerate bilinear form and a Lie algebra with Lie brackets that keeps it invariant is called, in particular, a quadratic Lie algebra. The $\mathfrak{d}$ becomes quadratic Lie algebra naturally.

We can also the inverse of Drinfel'd double. Let $\mathfrak{p}$ is quadratic Lie algebra. If $\mathfrak{a}, \mathfrak{b}$ is sub algebra of $\mathfrak{p}$ and isotropic for the bilinear form for $\mathfrak{p}$, i.e. $(x, y) = 0$ for arbitary $x, y \in \mathfrak{g}$, the $(\mathfrak{p}, \mathfrak{a}, \mathfrak{b})$ is called *Manin triple* [107]. The $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ becomes Manin triple.

3.3 Lie algebroid and Lie bialgebroid

The procedure Drinfel'd double for Lie algebroids in the previous section can be extended directly to Drinfel'd double for Lie algebroids. There are various types of algebroid structures, of which the Lie algebroid is the most basic. It is defined as follows..

Definition 3.3.1. The combination of the following structures $(E, [\cdot,\cdot]_E, \rho)$ is called a Lie algebroid.

- a vector bundle E over a manifold M
- a Liebracket for $\Gamma(E)$ (a section of E), $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$.
- a bundle map $\rho : E \to TM$.
  ρ and $[\cdot,\cdot]_E$ are satisfy following condition,
  $$\rho([X, Y]_E) = [\rho(X), \rho(Y)], \quad X, Y \in \Gamma(E). \tag{3.11}$$
  ρ is called anchor map.

Lie bracket $[\cdot,\cdot]_E$ is satisfies the Jacobi identity. The following condition is also satisfied,

$$[X, fY]_E = (\rho(X) \cdot f)Y + f[X, Y]_E. \tag{3.12}$$

for $f \in C^\infty(M)$. Here, $(\rho(X) \cdot f)$ means $\rho(X)$ acts as the differential operator for f.

Given a Lie algebroid, we can define the dual Lie algebroid $(E^*, [\cdot,\cdot]_{E^*}, \rho_*)$ on the same base manifold. Again, there is a natural inner product $\langle \cdot,\cdot \rangle$ between E and E^*. As a generalization of ordinary calculus for (multi)vectors and forms in $\Gamma(TM)$ and $\Gamma(T^*M)$, we define exterior algebras in $\Gamma(\wedge^\bullet E)$ and $\Gamma(\wedge^\bullet E^*)$. A natural inner product $\langle \xi, X \rangle$ between $\wedge^p E$ and $\wedge^p E^*$ is defined. We then define a Lie algebroid differential as a map $\mathrm{d} : \Omega_p(E) = \Gamma(\wedge^p E^*) \to \Omega_{p+1}(E)$ where $\Omega_p(E)$ is a generalization of p-form on T^*M. More explicitly, the exterior derivative d is defined through the action of $\xi \in \Gamma(\wedge^p E^*)$ on vectors $X_i \in \Gamma(E)$ [108]:

$$\begin{aligned} \mathrm{d}\xi(X_1, \ldots, X_{p+1}) = & \sum_{i=1}^{p+1} (-)^{i+1} \rho(X_i) \cdot \big(\xi(X_1, \ldots, \check{X}_i, \ldots, X_{p+1})\big) \\ & + \sum_{i<j} (-)^{i+j} \xi([X_i, X_j]_E, X_1, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X_{p+1}), \end{aligned} \tag{3.13}$$

where the notation $\check{X}_i$ stands for that the term is omitted in the expression. We sometimes use the notation such as $\xi(X) = \langle \xi, X \rangle$ for the natural scalar product between $\xi \in \Omega_p(E)$ and $X \in \Omega_p(E^*)$.

The exterior derivative, in particular, satisfies the following properties:

$$\begin{aligned} &\mathrm{d}(\xi \wedge \eta) = \mathrm{d}\xi \wedge \eta + (-)^{|\xi|}\xi \wedge \mathrm{d}\eta, \\ &\mathrm{d}f(X) = \rho(X) \cdot f, \\ &\mathrm{d}\xi(X, Y) = \rho(X) \cdot (\xi(Y)) - \rho(Y) \cdot (\xi(X)) - \xi([X, Y]_E), \end{aligned} \tag{3.14}$$

where $X, Y \in \Gamma(E)$, $\xi, \eta \in \Gamma(E^*)$. Similarly, a Lie derivative $\mathcal{L}_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^p E^*)$ by $X \in \Gamma(E)$ is defined by

$$\mathcal{L}_X(\xi)(Y_1, \ldots, Y_p) = \rho(X) \cdot (\xi(Y_1, \ldots, Y_p)) - \sum_{i=1}^{p} \xi\left(Y_1, \ldots, [X, Y_i]_E, \ldots, Y_p\right), \tag{3.15}$$

where $Y_1, \ldots, Y_p \in \Gamma(E)$, $\xi \in \Gamma(\wedge^p E^*)$. The interior product $\iota_X : \Gamma(\wedge^p E^*) \to \Gamma(\wedge^{p-1} E^*)$ by $X \in \Gamma(E)$ is defined by

$$(\iota_X \xi)(Y_1, \ldots, Y_{p-1}) = \xi(X, Y_1, \ldots, Y_{p-1}), \tag{3.16}$$

where $Y_1, \ldots, Y_{p-1} \in \Gamma(E), \xi \in \Gamma(\wedge^p E^*)$. They satisfy the following relations:

$$\begin{aligned} &\mathcal{L}_{[X,Y]_E} = \mathcal{L}_X \cdot \mathcal{L}_Y - \mathcal{L}_Y \cdot \mathcal{L}_X, \\ &\iota_{[X,Y]_E} = \mathcal{L}_X \cdot \iota_Y - \iota_Y \cdot \mathcal{L}_X, \\ &\mathcal{L}_X = \mathrm{d} \cdot \iota_X + \iota_X \cdot \mathrm{d}, \\ &\mathcal{L}_{fX}(\xi) = f\mathcal{L}_X(\xi) + \mathrm{d}f \wedge \iota_X(\xi), \end{aligned} \tag{3.17}$$

where $X, Y \in \Gamma(E)$, $f \in C^\infty(M)$, $\xi \in \Gamma(\wedge^\bullet E^*)$.

As we have discussed in the previous subsection, the Lie bracket $[\cdot, \cdot]_E$ can be generalized to those for multi-vectors $\Gamma(\wedge^p E)$. For $X \in \Gamma(\wedge^{p+1} E), Y \in \Gamma(\wedge^{q+1} E)$ and $f \in C^\infty(M)$, the Schouten-Nijenhuis bracket satisfies the following properties:

(i) $[X, Y]_{\mathrm{S}} = -(-)^{pq}[Y, X]_{\mathrm{S}}$.

(ii) $[X, f]_{\mathrm{S}} = \rho(X) \cdot f$ for $X \in \Gamma(E)$.

(iii) For $X \in \Gamma(\wedge^{p+1} E)$, the bracket $[X, \cdot]_{\mathrm{S}}$ acts on $\Gamma(\wedge^q E)$ as a degree-p derivation.

Here a derivation $\mathbf{D}$ is defined by an operator that satisfies the Leibniz rule $\mathbf{D}(ab) = \mathbf{D}a \cdot b + a\mathbf{D}b$. We also define an exterior derivative d_*, the interior product and the Lie derivative on $\Gamma(\wedge^\bullet E)$. We note that when the base manifold M consists of a point, then $\Gamma(E)$ represents a globally defined vector. In this case, $(E, [\cdot, \cdot]_E, \rho = 0)$ becomes a Lie algebra. We also note that $E = TM$, $\rho = \mathrm{id}$, $[X, Y]_E = \mathcal{L}_X Y$ defines a Lie algebroid with a trivial structure.

Once a Lie algebroid is defined, we can define a *Lie bialgebroid*. This is a generalization of the Lie bialgebra discussed in the previous subsection. Let $(E, [\cdot,\cdot]_E, \rho)$ be a Lie algebroid $E \xrightarrow{\pi} M$ and $(E^*, [\cdot,\cdot]_{E^*}, \rho_*)$ be its dual. For $X, Y \in \Gamma(\wedge^\bullet E)$ and $\mathrm{d}_* : \Gamma(\wedge^\bullet E) \to \Gamma(\wedge^{\bullet+1} E)$, if the following compatibility condition

$$\mathrm{d}_*[X,Y]_\mathrm{S} = [\mathrm{d}_* X, Y]_\mathrm{S} + [X, \mathrm{d}_* Y]_\mathrm{S} \tag{3.18}$$

is satisfied, then (E, E^*) is called a Lie bialgebroid over M. This implies that d_* acts on the Schouten-Nijenhuis bracket of $\Gamma(\wedge^\bullet E)$ as a derivation. Therefore we call (3.18) the derivation condition. The notion of a Lie bialgebroid was first introduced in [108]. If M is a point and ρ is trivial, then (E, E^*) becomes a Lie bialgebra and the condition (3.18) becomes the 1-cocycle condition (3.5).

3.4 Drinfel'd double for Lie bialgebroid and Courant algebroid

Now we consider the Drinfel'd double of a Lie bialgebroid (E, E^*). We may expect, from the discussion on the Lie bialgebra, that a double $E \oplus E^*$ possesses a Lie algebroid structure. However, the result is not the case. Before discussing this issue, we introduce the notion of *Courant algebroids* [102, 109]. Let $\mathcal{C} \xrightarrow{\pi} M$ be a vector bundle over M.

Definition 3.4.1. Courant algebroid is defined by the combination of the following structures $(\mathcal{C}, [\cdot,\cdot]_\mathrm{c}, \rho, (\cdot,\cdot))$ and following axioms C1-C5 [102].

- a vector bundle $\mathcal{C}$ on a manifold M.
- a Courant bracket $[\cdot,\cdot]_\mathrm{c} : \Gamma(\mathcal{C}) \times \Gamma(\mathcal{C}) \to \Gamma(\mathcal{C})$, this is an anti-symmetric bracket.
- an anchor map $\rho : \mathcal{C} \to TM$.
- an non-degenerate symmetric bilinear form $(\cdot,\cdot)$.

Axiom C1. For any $e_1, e_2, e_3 \in \Gamma(\mathcal{C})$, the Jacobiator of $[\cdot,\cdot]_\mathrm{c}$ is given by

$$[[e_1, e_2]_\mathrm{c}, e_3]_\mathrm{c} + \text{c.p.} = \mathcal{D}T(e_1, e_2, e_3), \tag{3.19}$$

where $T(e_1, e_2, e_3) = \frac{1}{3}([e_1, e_2]_\mathrm{c}, e_3) + \text{c.p.}$ and c.p. is terms obtained by the cyclic permutations.

Axiom C2. For any $e_1, e_2 \in \Gamma(\mathcal{C})$,

$$\rho_\mathrm{c}([e_1, e_2]_\mathrm{c}) = [\rho_\mathrm{c}(e_1), \rho_\mathrm{c}(e_2)], \tag{3.20}$$

where $[\cdot,\cdot]$ is the Lie bracket on TM.

Axiom C3. For any $e_1, e_2 \in \Gamma(\mathcal{C})$, $f \in C^\infty(M)$,

$$[e_1, f e_2]_\mathrm{c} = f[e_1, e_2]_\mathrm{c} + (\rho_\mathrm{c}(e_1) \cdot f) e_2 - (e_1, e_2)\mathcal{D}f. \tag{3.21}$$

Axiom C4. $\rho_c \cdot \mathcal{D} = 0$, namely, for any $f, g \in C^\infty(M)$, we have

$$(\mathcal{D}f, \mathcal{D}g) = 0. \tag{3.22}$$

Axiom C5. For any $e_1, e_2, e_3 \in \Gamma(\mathcal{C})$, we have the compatibility between the bilinear form $(\cdot, \cdot)$ and the anchor ρ_c:

$$\rho_c(e_1) \cdot (e_2, e_3) = ([e_1, e_2]_c + \mathcal{D}(e_1, e_2), e_3) + (e_2, [e_1, e_3]_c + \mathcal{D}(e_1, e_3)). \tag{3.23}$$

Here, $\mathcal{D} = \frac{1}{2}\beta^{-1*}\rho_c \mathrm{d}_0$, in which d_0 is a natural exterior derivative on T^*M, such that $(\mathcal{D}f, e) = \frac{1}{2}\rho_c(e) \cdot f$ for $f \in C^\infty(M)$, $e \in \Gamma(\mathcal{C})$. Then $(\mathcal{C}, [\cdot, \cdot]_c, \rho_c, (\cdot, \cdot))$ defines a Courant algebroid. We note that axioms C1-C5 are not independent. Indeed, it is shown that the axioms C3 and C4 follow from C5 and C2 respectively [110]. There is a definition of Courant algebroids based on a non-skew-symmetric bracket for which the Jacobi identity holds. In the following, we employ the definition based on a skew-symmetric bracket.

Instead of the above-mentioned antisymmetric Courant bracket $[\cdot, \cdot]_c$, there is also a way to define it using the binary operation $\circ$ [111,112]. In this case, Axiom C1-C5 shown above is replaced by Axiom C1′-C5′ as follows.

$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) \tag{Axiom C1′}$$

$$\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)] \tag{Axiom C2′}$$

$$e_1 \circ (f e_2) = f(e_1 \circ e_2) + (\rho(e_1) f) e_2 \tag{Axiom C3′}$$

$$e_1 \circ e_1 = \mathcal{D}(e_1, e_1) \tag{Axiom C4′}$$

$$\rho(e_3)(e_1, e_2) = (e_3 \circ e_1, e_2) + (e_1, e_3 \circ e_2) \tag{Axiom C5′}$$

The equivalence of these definition is proved in [111]. As can be seen from Axiom C1 ′, $\circ$ satisfies the left Leibniz identity instead of the Jacobi identity. From the definition by the antisymmetric bracket $[\cdot, \cdot]_c$, the Courant algebroid can be interpreted as a strong Homotopy Lie algebra [113]. On the other hand, from the definition by the binary operation $\circ$, the Courant algebroid can also be interpreted as Leibnitz algebroid.

Now we discuss the double of a Lie bialgebroid (E, E^*). This notion was first introduced by Liu, Weinstein and Xu in [102]. Given a Lie bialgebroid (E, E^*), they considered the following doubled structure on $\mathcal{C} = E \oplus E^*$:

(I) For $X_1, X_2 \in \Gamma(E), \xi_1, \xi_2 \in \Gamma(E^*)$, a non-degenerate, bilinear forms $(\cdot, \cdot)_\pm$ are defined by

$$(X_1 + \xi_1, X_2 + \xi_2)_\pm = \frac{1}{2}\Big\{ \langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle \Big\}, \tag{3.24}$$

where $\langle \cdot, \cdot \rangle$ is a natural inner product between E and E^*

(II) A skew-symmetric bracket $[\cdot, \cdot]_c$ on $\Gamma(\mathcal{C})$ is defined by

$$\begin{aligned} [e_1, e_2]_c = {} & [X_1, X_2]_E + \mathcal{L}_{\xi_1} X_2 - \mathcal{L}_{\xi_2} X_1 - \mathrm{d}_*(e_1, e_2)_- \\ & + [\xi_1, \xi_2]_{E^*} + \mathcal{L}_{X_1} \xi_2 - \mathcal{L}_{X_2} \xi_1 + \mathrm{d}(e_1, e_2)_-, \end{aligned} \tag{3.25}$$

where $e_i = X_i + \xi_i \in \Gamma(\mathcal{C})$, $(i = 1, 2)$.

(III) An anchor $\rho_{\rm c} : \mathcal{C} \to TM$ is defined by $\rho_{\rm c} = \rho + \rho_*$. Namely, $\rho_{\rm c}(X + \xi) = \rho(X) + \rho_*(\xi)$ for $^\forall X \in \Gamma(E)$, $^\forall \xi \in \Gamma(E^*)$.

(IV) An exterior derivative $\mathcal{D} = {\rm d} + {\rm d}_*$ on $\mathcal{C} = E \oplus E^*$ is defined.

Given these structures, the authors in [102] showed that $(\mathcal{C} = E \oplus E^*, [\cdot,\cdot]_{\rm c}, \rho_{\rm c}, (\cdot,\cdot)_+)$ becomes a Courant algebroid satisfying the axioms C1-C5. We stress that the Jacobiator of the Courant bracket $[\cdot,\cdot]_{\rm c}$ does not vanish in general. Therefore a Courant algebroid, obtained by the double of a Lie bialgebroid, is not a Lie algebroid. This is in contrast to the double of a Lie bialgebra.

The authors in [102] also showed that if there are complementally isotropic subbundles E, E^* with respect to the bilinear product $(\cdot,\cdot)$ in a Courant algebroid $\mathcal{C}$, and if they are closed under the Courant bracket $[\cdot,\cdot]_{\rm c}$, then there is a natural Lie bialgebroid structure on (E, E^*). When E, E^* are maximally isotropic, *i.e.* $\dim E = \dim E^* = \frac{1}{2}\dim \mathcal{C}$, then E, E^* are called Dirac structures and they provide therefore a natural generalization, a Lie algebroid analogue, of the Manin triple $(\mathcal{C}, E, E^*)$. Indeed, when M consists of a point, a Courant algebroid becomes a quadratic Lie algebra, namely, a Lie algebra with the non-degenerate bilinear form $(\cdot,\cdot)$. This is just the double of a Lie bialgebra.

The Courant bracket naturally appears in the context of generalized geometry [37] where the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ is prepared in order to realize manifest T-duality. We note that the original Courant bracket on $TM \oplus T^*M$ introduced by T. Courant is defined by

$$[X_1 + \xi_1, X_2 + \xi_2]_{\rm c} = [X_1, X_2] + (\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1) + \frac{1}{2}{\rm d}_0(\xi_1(X_2) - \xi_2(X_1)), \tag{3.26}$$

for $X_i \in \Gamma(TM)$, $\xi_i \in \Gamma(T^*M)$. We call (3.26) the c-bracket. It of course satisfies the axioms C1-C5. We will comment on the relations of DFT and generalized geometry in Section 5.3.

A few comments are in order. First, it is not always true that a Courant algebroid is defined by Lie bialgebroids [111]. Second, we consider a class of Courant algebroids called exact [109] in the following.

3.5 Vaisman algebroid and doubled structure

In this section, we study doubled aspects of Vaisman algebroids. It has been discussed that the gauge symmetry algebra of DFT, which is governed by the C-bracket, is characterized by an algebroid proposed by Vaisman [72, 73]. We call this the Vaisman algebroid. In the following, we introduce the notion of the Vaisman algebroid and discuss its doubled structures.

The definition of the Vaisman algebroid is the following. Let $\mathcal{V} \xrightarrow{\pi} M$ be a vector bundle over a manifold M. We introduce a non-degenerate symmetric bilinear form $(\cdot,\cdot) : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to M \times \mathbb{C}$ and an anchor $\rho_{\rm V} : \mathcal{V} \to TM$. A map $\mathcal{D} : C^\infty(M) \to \Gamma(\mathcal{V})$ is defined through $(\mathcal{D}f, e) = \frac{1}{2}\rho_{\rm V}(e) \cdot f$ for $e \in \Gamma(\mathcal{V})$ and $f \in C^\infty(M)$. When a skew-symmetric bracket

$[\cdot,\cdot]_{\mathrm{V}}$ on $\Gamma(\mathcal{V})$ satisfies the following axioms V1-V2, then a Vaisman algebroid is defined by $(\mathcal{V},[\cdot,\cdot]_{\mathrm{V}},\rho_{\mathrm{V}},(\cdot,\cdot))$:

Definition 3.5.1. Vaisman algebroid is defined by the combination of structures $(\mathcal{V},[\cdot,\cdot]_{\mathrm{V}},\rho_{\mathrm{V}},(\cdot,\cdot))$ and Axiom V1, V2.

- a vector bundle $\mathcal{V}$ over a manifold M. $\mathcal{V}\xrightarrow{\pi}M$
- Vaisman bracket $[\cdot,\cdot]_{\mathrm{V}}:\Gamma(\mathcal{V})\times\Gamma(\mathcal{V})\to\Gamma(\mathcal{V})$, this is an anti-symmetric bracket.
- an anchor map $\rho_{\mathrm{V}}:\mathcal{V}\to TM$.
- a non-degenerate symmetric bilinear form$(\cdot,\cdot)$

Axiom V1. $[e_1,fe_2]_{\mathrm{V}}=f[e_1,e_2]_{\mathrm{V}}+(\rho_{\mathrm{V}}(e_1)\cdot f)e_2-(e_1,e_2)\mathcal{D}f$.

Axiom V2. $\rho_{\mathrm{V}}(e_1)\cdot(e_2,e_3)=([e_1,e_2]_{\mathrm{V}}+\mathcal{D}(e_1,e_2),e_3)+(e_2,[e_1,e_3]_{\mathrm{V}}+\mathcal{D}(e_1,e_3))$.

Here $e_1,e_2\in\Gamma(\mathcal{V})$, $f\in C^\infty(M)$, and $\mathcal{D}:C^\infty(M)\to\Gamma(\mathcal{V})$ is defined through $(\mathcal{D}f,e)=\frac{1}{2}\rho_{\mathrm{V}}(e)\cdot f$ for $e\in\Gamma(\mathcal{V})$ and $f\in C^\infty(M)$.

Compared to the definition of Courant algebroid, Axiom V1 corresponds to Axiom C3 and Axiom V2 to Axiom C5. The Vaisman algebroid is therefore a further generalization of the Courant algebroid. It is also consistent with [110].

Following the prescription given by Liu-Weinstein-Xu [102] on Courant algebroids, we examine doubled aspects of Vaisman algebroids. Given a Lie algebroid $(E,[\cdot,\cdot]_E,\rho)$ and its dual $(E^*,[\cdot,\cdot]_{E^*},\rho_*)$ over a manifold M, we consider a vector bundle $\mathcal{V}=E\oplus E^*$. Note that we never assume the Lie bialgebroid structures on (E,E^*). We then define non-degenerate bilinear forms $(\cdot,\cdot)_\pm$ on $\mathcal{V}$ as

$$(e_1,e_2)_\pm=\frac{1}{2}\Big(\langle\xi_1,X_2\rangle\pm\langle\xi_2,X_1\rangle\Big),\tag{3.27}$$

where $e_i=X_i+\xi_i\in\Gamma(\mathcal{V})$ $(i=1,2)$, $X_i\in\Gamma(E),\xi_i\in\Gamma(E^*)$ and $\langle\cdot,\cdot\rangle$ is the inner product between E and E^*. We next define a skew-symmetric bracket $[\cdot,\cdot]_{\mathrm{V}}$ in $\Gamma(\mathcal{V})$ as

$$\begin{aligned}[e_1,e_2]_{\mathrm{V}}=[X_1,X_2]_E&+\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1-\mathrm{d}_*(e_1,e_2)_-\\&+[\xi_1,\xi_2]_{E^*}+\mathcal{L}_{X_1}\xi_2-\mathcal{L}_{X_2}\xi_1+\mathrm{d}(e_1,e_2)_-,\end{aligned}\tag{3.28}$$

where $\mathcal{L}_X,\mathcal{L}_\xi,\mathrm{d}_*,\mathrm{d}$ are natural Lie derivatives and de Rham differentials defined on $\Gamma(E),\Gamma(E^*)$. We employ the morphism $\rho_{\mathrm{V}}=\rho+\rho_*$ as the anchor in $\mathcal{V}$. The map $\mathcal{D}:C^\infty(M)\to\Gamma(\mathcal{V})$ is defined by $(\mathcal{D}f,e)=\frac{1}{2}\rho_{\mathrm{V}}(e)\cdot f$ which is expressed as $\mathcal{D}=\mathrm{d}+\mathrm{d}_*$. The expression (3.28) is nothing but the one defined in (3.25) but we here again stress that we never assume that (E,E^*) is a Lie bialgebroid, *i.e.* the derivation condition (3.18) is not satisfied in general. In the following, we show that $(E\oplus E^*,\rho+\rho_*,[\cdot,\cdot]_{\mathrm{V}},(\cdot,\cdot)_+)$ introduced above indeed defines a Vaisman algebroid, but not a Courant algebroid.

We now examine to what extent the axioms C1-C5 for the Courant algebroid are lacked due to the failure of the derivation condition. The auxiliary relations needed in the process of proof are the following two equations.

$$\begin{aligned} T(e_1, e_2, e_3) &\equiv \frac{1}{3}(([e_1, e_2]_{\mathrm{c}}, e_3)_+ + \text{c.p.}) \\ &= \frac{1}{2}\langle [X_1, X_2]_E, \xi_3\rangle + \langle [\xi_1, \xi_2]_{E^*}, X_3 + \rho(X_3)(e_1, e_2)_- - \rho_*(\xi_3)(e_1, e_2)_-\rangle, \end{aligned} \tag{3.29}$$

$$\begin{aligned} ([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.} &= T(e_1, e_2, e_3) \\ &\quad + [\{\rho(X_3)(e_1, e_2)_- + 2\rho_*(\xi_3)(e_1, e_2)_- - \langle [\xi_1, \xi_2]_{E^*}, X_3\rangle\} + \text{c.p.}]. \end{aligned} \tag{3.30}$$

For detailed calculations, see Appendix A.3-A.7.

Axiom C1

First, we check the Jacobiator of the Vaisman bracket $[\cdot, \cdot]_{\mathrm{V}}$. Expanding the left-hand side of the (3.19), we obtain the following result.

$$[[e_1, e_2]_{\mathrm{V}}, e_3]_{\mathrm{V}} + \text{c.p.} = I_1 + I_2, \tag{3.31}$$

$$\begin{aligned} I_1 = {} & [[\xi_1, \xi_2]_{E^*}, \xi_3]_{E^*} + [\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1, \xi_3]_{E^*} + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} \\ & + \mathcal{L}_{[X_1, X_2]_E + \mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1 - \mathrm{d}_*(e_1, e_2)_-}\xi_3 \\ & - \mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{X_3}\mathcal{L}_{X_1}\xi_2 + \mathcal{L}_{X_3}\mathcal{L}_{X_2}\xi_1 - \mathcal{L}_{X_3}\mathrm{d}(e_1, e_2)_- + \mathrm{d}([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} I_2 = {} & [[X_1, X_2]_E, X_3]_E + [\mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1, X_3]_E - [\mathrm{d}_*(e_1, e_2)_-, X_3]_E \\ & + \mathcal{L}_{[\xi_1, \xi_2]_{E^*} + \mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1 + \mathrm{d}(e_1, e_2)_-}X_3 \\ & - \mathcal{L}_{\xi_3}[X_1, X_2]_E - \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_1}X_2 + \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_2}X_1 + \mathcal{L}_{\xi_3}\mathrm{d}_*(e_1, e_2)_- - \mathrm{d}_*([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.} \end{aligned} \tag{3.33}$$

Here, terms in $\Gamma(E^*)$ are written as I_1, terms in $\Gamma(E)$ are written as I_2. Subsequent calculations are performed only with respect to I_1. By interchanging X and ξ, the calculations for I_2 can also be reproduced. The above I_1 contains terms that disappear due to the Jacobiator of the Lie bracket. The following form can be obtained by rearranging them.

$$\begin{aligned} I_1 = {} & [\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1, \xi_3]_{E^*} + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} + \mathcal{L}_{\mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1}\xi_3 - \mathcal{L}_{\mathrm{d}_*(e_1, e_2)_-}\xi_3 \\ & - \mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{X_3}\mathrm{d}(e_1, e_2)_- + \mathrm{d}([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.} \end{aligned} \tag{3.34}$$

The following formula is applied to I_1. The derivation of this formula is given in Appendix A.3.2

$$\begin{aligned} \mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*} + \text{c.p.} = {} & [\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1, \xi_3]_{E^*} + \mathcal{L}_{\mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1}\xi_3 \\ & + 2[\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} + 2\mathrm{d}(\rho_*(\xi_3)\cdot(e_1, e_2)_-) - \mathrm{d}\langle [\xi_1, \xi_2]_{E^*}, X_3\rangle \\ & + \iota_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{\mathrm{V}} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1) + \text{c.p.} \end{aligned} \tag{3.35}$$

Then, finally, I_1 can be summarised as follows.

$$I_1 = \mathrm{d}T(e_1, e_2, e_3) - \{K_1 + K_2\} + \text{c.p.} \tag{3.36}$$

Here,

$$\begin{aligned} K_1 &= \iota_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1), \\ K_2 &= \mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}\xi_3 + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*}. \end{aligned} \tag{3.37}$$

Similarly, calculations with respect to I_2, we obtainy the following results.

$$\begin{aligned} I_2 &= \mathrm{d}_*T(e_1, e_2, e_3) - \{K_3 + K_4\} + \text{c.p}, \\ K_3 &= \iota_{\xi_3}(\mathrm{d}_*[X_1, X_2]_E - \mathcal{L}_{X_1}\mathrm{d}_*X_2 + \mathcal{L}_{X_2}\mathrm{d}_*X_1), \\ K_4 &= -\Big(\mathcal{L}_{\mathrm{d}(e_1,e_2)_-}X_3 + [\mathrm{d}_*(e_1, e_2)_-, X_3]_E\Big). \end{aligned} \tag{3.38}$$

Thus, the Jacobiator of the Vaisman bracket $[\cdot, \cdot]_{\mathrm{V}}$ becomes

$$[[e_1, e_2]_{\mathrm{V}}, e_3]_{\mathrm{V}} + \text{c.p.} = I_1 + I_2 = \mathcal{D}T(e_1, e_2, e_3) - (J_1 + J_2 + \text{c.p.}). \tag{3.39}$$

where

$$\begin{aligned} J_1 &= K_1 + K_3 \\ &= i_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1) + i_{\xi_3}(\mathrm{d}_*[X_1, X_2]_E - \mathcal{L}_{X_1}\mathrm{d}_*X_2 + \mathcal{L}_{X_2}\mathrm{d}_*X_1), \\ J_2 &= K_2 + K_4 \\ &= \mathcal{L}_{\mathrm{d}(e_1,e_2)_-}\xi_3 + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} + \mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}X_3 + [\mathrm{d}_*(e_1, e_2)_-, X_3]_E. \end{aligned} \tag{3.40}$$

In general, for any X_i, ξ_i, f, $(J_1 + J_2 + \text{c.p.})$is not 0. Therefore, Axiom C1 is broken in $(\mathcal{V}, [\cdot, \cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot, \cdot)_+)$.

Axiom C2

Similarly, we check the Axiom C2. First, each side of (3.20) act for f respectively, we obtain

$$\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}}) \cdot f = [\rho_{\mathrm{V}}(e_1), \rho_{\mathrm{V}}(e_2)]f. \tag{3.41}$$

The left-hand side of the equation (3.41) expands as follows.

$$\begin{aligned} &\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}}) \cdot f \\ &= [\rho(X_1), \rho(X_2)] \cdot f + \rho(\mathcal{L}_{\xi_1}X_2) \cdot f - \rho(\mathcal{L}_{\xi_2}X_1) \cdot f - \frac{1}{2}\rho\rho_*^*\mathrm{d}_0(\langle\xi_1, X_2\rangle - \langle\xi_2, X_1\rangle) \cdot f \\ &+ [\rho_*(\xi_1), \rho_*(\xi_2)] \cdot f + \rho_*(\mathcal{L}_{X_1}\xi_2) \cdot f - \rho_*(\mathcal{L}_{X_2}\xi_1) \cdot f + \frac{1}{2}\rho_*\rho^*\mathrm{d}_0(\langle\xi_1, X_2\rangle - \langle\xi_2, X_1\rangle) \cdot f, \end{aligned} \tag{3.42}$$

where $\mathrm{d}_* = \rho_*^*\mathrm{d}_0, \mathrm{d} = \rho^*\mathrm{d}_0. \mathrm{d}_0$ is the differential operator on $\Gamma(T^*M)$.

The right-hand side of the equation (3.41) expands as follows.

$$\begin{aligned}&[\rho_{\mathrm{V}}(e_1),\rho_{\mathrm{V}}(e_2)]f\\&=[\rho(X_1),\rho(X_2)]f+[\rho_*(\xi_1),\rho(X_2)]f+[\rho(X_1),\rho_*(\xi_2)]f+[\rho_*(\xi_1),\rho_*(\xi_2)]f.\end{aligned}\tag{3.43}$$

The closs terms for the two and three X,ξ entries on the right-hand side become

$$\begin{aligned}[\rho(X),\rho_*(\xi)]f&=(\rho(X)\rho_*(\xi)-\rho_*(\xi)\rho(X))\cdot f\\&=\rho(X)\rho_*(\xi)\cdot f+(\rho\rho_*^*\mathrm{d}_0\langle\xi,X\rangle)\cdot f-\langle\xi,\mathcal{L}_{\mathrm{d}f}X\rangle-\rho(\mathcal{L}_\xi X)\cdot f.\end{aligned}\tag{3.44}$$

However, the following formulae are used

$$\rho_*(\xi)\rho(X)\cdot f=-\rho\rho_*^*\mathrm{d}_0\langle\xi,X\rangle\cdot f+\langle\xi,\mathcal{L}_{\mathrm{d}f}X\rangle+\langle\mathrm{d}f,\mathcal{L}_\xi X\rangle.\tag{3.45}$$

Therefore, the difference between the left and right sides of (3.41) is

$$\begin{aligned}&\rho_{\mathrm{V}}([e_1,e_2]_{\mathrm{V}})\cdot f-[\rho_{\mathrm{V}}(e_1),\rho_{\mathrm{V}}(e_2)]f\\&=-\langle\xi_1,\big(\mathcal{L}_{\mathrm{d}f}X_2-[X_2,\mathrm{d}_*f]_E\big)\rangle+\langle\xi_2,\big(\mathcal{L}_{\mathrm{d}f}X_1-[X_1,\mathrm{d}_*f]_E\big)\rangle\\&\quad+\frac{1}{2}\Big(\rho\rho_*^*+\rho_*\rho^*\Big)\mathrm{d}_0(\langle\xi_1,X_2\rangle-\langle\xi_2,X_1\rangle)\cdot f.\end{aligned}\tag{3.46}$$

In general, the right-hand side is not 0 for any X_i,ξ_i. Therefore, Axiom C2 is broken in $(\mathcal{V},[\cdot,\cdot]_{\mathrm{V}},\rho_{\mathrm{V}},(\cdot,\cdot)_+)$.

Axiom C3

Next, we check Axiom C3. Using $e_i=X_i+\xi_i$,the left-hand side of the (3.21) can be written as

$$[e_1,fe_2]_{\mathrm{V}}=[X_1,fX_2]_{\mathrm{V}}+[X_1,f\xi_2]_{\mathrm{V}}+[\xi_1,fX_2]_{\mathrm{V}}+[\xi_1,f\xi_2]_{\mathrm{V}}.\tag{3.47}$$

Here, from the definition of the Vaisman bracket, the right-hand side of (3.47) expands as follows.

$$\begin{aligned}&[X_1,fX_2]_{\mathrm{V}}=[X_1,fX_2]_E,\\&[X_1,f\xi_2]_{\mathrm{V}}=f[X_1,\xi_2]_{\mathrm{V}}+(\rho(X_1)\cdot f)\xi_2-\frac{1}{2}\mathcal{D}f\langle\xi_2,X_1\rangle,\\&[\xi_1,fX_2]_{\mathrm{V}}=f[\xi_1,X_2]_{\mathrm{V}}+(\rho_*(\xi_1)\cdot f)X_2-\frac{1}{2}\mathcal{D}f\langle\xi_1,X_2\rangle,\\&[\xi_1,f\xi_2]_{\mathrm{V}}=[\xi_1,f\xi_2]_{E^*}.\end{aligned}\tag{3.48}$$

Adding up all the right-hand sides of the four equations in (A.57) yields the following result.

$$[e_1,fe_2]_{\mathrm{V}}=f[e_1,e_2]_{\mathrm{V}}+(\rho(e_1)f)e_2-\mathcal{D}f(e_1,e_2)_+.\tag{3.49}$$

Therefore, Axiom C3 holds in $(\mathcal{V},[\cdot,\cdot]_{\mathrm{V}},\rho_{\mathrm{V}},(\cdot,\cdot)_+)$.

Axiom C4

Next, we check Axiom C4. The left-hand side of the (3.22) equation expands as follows.

$$(\mathcal{D}f, \mathcal{D}g)_+ = \frac{1}{2}(\rho_* \rho^* \mathrm{d}_0 f + \rho \rho_*^* \mathrm{d}_0 f) g. \tag{3.50}$$

In general, the anchor ρ is not antisymmetric. That is, $\rho\rho_*^* = -\rho_*\rho^*$ does not hold. Therefore, the right-hand side of (3.50) does not equal 0 and Axiom C4 is broken at $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$. The superscript $*$ attached to the anchor means adjoint operator, which is defined by the transpose of the original operator through the inner product.

$$\begin{aligned} &\rho : E \to TM \qquad \rho^* : T^*M \to E^* \\ &\rho_* : E^* \to TM \quad \rho_*^* : T^*M \to E \end{aligned} \tag{3.51}$$

Thus, $\rho\rho_*^* : T^*M \to TM$ and $\rho_*\rho^* : T^*M \to TM$.

As an aside, If derivation condition is satisfied, the anchor becomes antisymmetric. This is shown in the Proposition 3.4 of [108]. In general, the properties of Lie subalgebras yield the following relations.

$$\begin{aligned} &(\mathcal{L}_{\mathrm{d}f} X + [\mathrm{d}_* f, X]_E) \wedge Y \\ &= -f\Big(\mathrm{d}_*[X,Y]_E + \mathcal{L}_Y \mathrm{d}_* X - \mathcal{L}_X \mathrm{d}_* Y\Big) + \Big(\mathrm{d}_*[X, fY]_E - \mathcal{L}_X \mathrm{d}_*(fY) + \mathcal{L}_{fY}\mathrm{d}_* X\Big). \end{aligned} \tag{3.52}$$

Proposition 3.4 shows that the right-hand side of (3.67) becomes 0 by imposing the derivation condition. In other words, it can be confirmed that when the derivation condition is satisfied, the following relation is established incidentally.

$$\mathcal{L}_{\mathrm{d}f} X + [\mathrm{d}_* f, X]_E = 0. \tag{3.53}$$

It can be shown that ρ is not antisymmetric unless (3.53) is satisfied. See Appendix A.6 for details. Therefore, if the derivation condition is satisfied, rho is always antisymmetric and Axiom C4 holds. Conversely, if the derivation condition is not imposed, the case where the anchor is not antisymmetric cannot be ruled out..

Axiom C5

Finally, we check the axiom C5. From the relation (3.29) for $T(e_1, e_2, e_3)$, the following two equations hold.

$$([e, e_1]_{\mathrm{V}}, e_2)_+ = T(e, e_1, e_2) + \frac{1}{2}\rho_{\mathrm{V}}(e) \cdot (e_1, e_2)_+ - \frac{1}{2}\rho_{\mathrm{V}}(e_1) \cdot (e, e_2)_+, \tag{3.54}$$

$$(e_1, [e, e_2]_{\mathrm{V}})_+ = T(e, e_2, e_1) + \frac{1}{2}\rho_{\mathrm{V}}(e) \cdot (e_2, e_1)_+ - \frac{1}{2}\rho_{\mathrm{V}}(e_2) \cdot (e, e_1)_+. \tag{3.55}$$

Adding the left and right hand sides of (3.54) and (3.55) respectively, we obtain,

$$\begin{aligned} &([e, e_1]_{\mathrm{V}}, e_2)_+ + (e_1, [e, e_2]_{\mathrm{V}})_+ \\ &= T(e, e_1, e_2) + \frac{1}{2}\rho_{\mathrm{V}}(e) \cdot (e_1, e_2)_+ - \frac{1}{2}\rho_{\mathrm{V}}(e_1) \cdot (e, e_2)_+ \\ &\quad + T(e, e_2, e_1) + \frac{1}{2}\rho_{\mathrm{V}}(e) \cdot (e_2, e_1)_+ - \frac{1}{2}\rho_{\mathrm{V}}(e_2) \cdot (e, e_1)_+. \end{aligned} \tag{3.56}$$

Using the antisymmetry of T, we can rewrite the (3.56) as

$$\rho_{\mathrm{V}}(e)\cdot(e_1,e_2)_+ = ([e,e_1]_{\mathrm{V}},e_2)_+ + (e_1,[e,e_2]_{\mathrm{V}})_+ + \frac{1}{2}\rho_{\mathrm{V}}(e_1)\cdot(e,e_2)_+ + \frac{1}{2}\rho_{\mathrm{V}}(e_2)\cdot(e,e_1)_+$$
$$([e,e_1]_{\mathrm{V}} + \mathcal{D}(e,e_1)_+,e_2)_+ + (e_1,[e,e_2]_{\mathrm{V}} + \mathcal{D}(e,e_2)_+)_+. \tag{3.57}$$

Therefore, Axiom C5 holds in $(\mathcal{V},[\cdot,\cdot]_{\mathrm{V}},\rho_{\mathrm{V}},(\cdot,\cdot)_+)$. The details of these calculations are given in Appendix A.5 and A.7. It can now be shown that the Vaisman algebroid is obtained by the double of two dual Lie algebroid. This is consistent with the discussion in [110].

Conversely, we consider the split of the Vaisman algebroid with the Dirac structure. It is shown that when there are Dirac subbundles $L,\tilde{L}$ of a Courant algebroid $(\mathcal{C},[\cdot,\cdot]_{\mathrm{c}},\rho_{\mathrm{c}},(\cdot,\cdot))$, namely, $L,\tilde{L}$ are maximally isotropic with respect to $(\cdot,\cdot)$, satisfing $\mathcal{C}=L\oplus\tilde{L}$ and involutive (integrable), then the vector bundle $\tilde{L}$ is regarded as the dual bundle of L under the natural paring $2(\cdot,\cdot)$. Given these structures, it is shown that $(L,\tilde{L})$ becomes a Lie bialgebroid. We briefly demonstrate this fact following the discussion in [102]. Before showing the above statement, we first refer the Proposition 2.3 in [102]:

Proposition 2.3 in [102]. If L is an integrable isotropic subbundle of a Courant algebroid $(\mathcal{C},[\cdot,\cdot]_{\mathrm{c}},\rho_{\mathrm{c}},(\cdot,\cdot))$, then $(L,[\cdot,\cdot]_{\mathrm{c}},\rho_{\mathrm{c}}|_L)$ becomes a Lie algebroid.

Here the isotropy is defined with respect to the bilinear form $(\cdot,\cdot)$. Namely, for any $X,Y\in\Gamma(L)$, they satisfy $(X,Y)=0$. This proposition is confirmed by showing that the bracket $[\cdot,\cdot]_{\mathrm{c}}$ on L satisfies the Jacobi identity. This immediately follows from the relation (3.19) in axiom C1 of Courant algebroids and the isotropic nature of L. Due to the proposition 2.3, any Dirac structures $L,\tilde{L}$ in a Courant algebroid become Lie algebroids. Their anchors are defined by $\rho=\rho_{\mathrm{c}}|_L$, $\rho_*=\rho_{\mathrm{c}}|_{\tilde{L}}$.

By its defining axiom C5, for $X\in\Gamma(L)$, $\xi\in\Gamma(\tilde{L})$, one can show that

$$[X,\xi]_{\mathrm{c}} = -\mathcal{L}_\xi X + \frac{1}{2}\mathrm{d}_*\langle\xi,X\rangle + \mathcal{L}_X\xi - \frac{1}{2}\mathrm{d}\langle\xi,X\rangle. \tag{3.58}$$

Here in deriving (3.58), we have used the fact that $L,\tilde{L}$ are Lie algebroids and isotropic. With this relation, the following Lemma 5.2 in [102] follows:

Lemma 5.2 in [102]. Given Dirac structures $L,\tilde{L}$ such that $\mathcal{C}=L\oplus\tilde{L}$ for a Courant algebroid $\mathcal{C}$, then the following relations hold:

$$\mathcal{L}_{\mathrm{d}_*f}\xi = -[\mathrm{d}f,\xi]_{\tilde{L}}, \quad \mathcal{L}_{\mathrm{d}f}X = -[\mathrm{d}_*f,X]_L. \tag{3.59}$$

Here d,d_* are induced de Rham differentials on L and $\tilde{L}$.

This is shown as follows. By the axiom C4, one first find the relation

$$\rho_*\cdot\mathrm{d} = -\rho\cdot\mathrm{d}_*. \tag{3.60}$$

Then using this relation, we find

$$[\rho_*(\xi),\rho(X)] = \rho(\mathcal{L}_\xi X) - \rho_*(\mathcal{L}_X\xi) + \rho_*(\mathrm{d}\langle\xi,X\rangle), \tag{3.61}$$

where we have assumed the axiom C2 and used the relation (3.60). On the other hand, by using the properties of Lie algebroids, we calculate

$$\rho_*(\mathrm{d}\langle\xi, X\rangle)\cdot f = [\rho_*(\xi), \rho(X)]f - \rho(\mathcal{L}_\xi X)\cdot f + \rho_*(\mathcal{L}_X\xi)\cdot f + \langle\mathcal{L}_{\mathrm{d}_*f}\xi + [\mathrm{d}f, \xi]_{\tilde{L}}, X\rangle. \quad (3.62)$$

comparing the above relations, one proofs the first part in (3.59). Performing the same calculus by exchanging $\xi \leftrightarrow X$ the latter also follows.

Given the Lemma 5.2, now we focus on the Jacobiator of the Courant bracket. As we have shown before, if $L, \tilde{L}$ are Lie algebroids, we have

$$[[e_1, e_2]_{\mathrm{c}}, e_2]_{\mathrm{c}} + \text{c.p.} = \mathcal{D}T(e_1, e_2, e_3) - (J_1 + J_2 + \text{c.p.}), \quad (3.63)$$

where J_1, J_2 are given in (3.40). Since $\mathcal{C} = L \oplus \tilde{L}$ satisfies the axiom C1, we have $J_1 + J_2 + \text{c.p.} = 0$. Due to the Lemma 5.2, one can show that $J_2 = 0$ and the above condition implies $J_1 + \text{c.p.} = 0$. If we take $e_1 = X_1$, $e_2 = X_2$, $e_3 = \xi_3$, then this condition yields

$$\mathrm{d}_*[X_1, X_2]_L - \mathcal{L}_{X_1}\mathrm{d}_*X_2 + \mathcal{L}_{X_2}\mathrm{d}_*X_1 = 0. \quad (3.64)$$

This is nothing but the derivation condition (3.18) for Lie bialgebroids. As we have mentioned before, Dirac structures $L, \tilde{L}$ in a Courant algebroid defines a Manin triple $(\mathcal{C}, L, \tilde{L})$.

We then in turn switch to the discussion on Vaisman algebroids. A Dirac structure on a Vaisman algebroid $\mathcal{V}$ is defined by a maximally isotropic subbundle in $\mathcal{V}$ with respect to a bilinear form $(\cdot, \cdot)$ defined on $\Gamma(\mathcal{V})$. Now we assume that there are Dirac structures $L, \tilde{L}$ such that $\mathcal{V} = L \oplus \tilde{L}$ in a Vaisman algebroid $\mathcal{V}$. Indeed, there is a Dirac structure in a Vaisman algebroid defined in a para-Kähler manifold [72, 73]. For Vaisman algebroids, however, only the axioms C3 and C5 of Courant algebroids are satisfied. Obviously, the proposition 2.3 in [102] does not follow since it requires the axiom C1. Therefore, the bracket does not satisfy the Jacobi identity and $L, \tilde{L}$ are not Lie algebroid in general. Even though they have Lie algebroid structures, since $\mathcal{V} = L \oplus \tilde{L}$ does not satisfy the axioms C2 and C4, Lemma 5.2 in [102] does not hold. Therefore we conclude that the Dirac structures $L, \tilde{L}$ in Vaisman algebroids do not satisfy the derivation condition and they never define a Lie bialgebroid in general. It is known that a Lie algebroid L and its dual L^* form a Lie bialgebroid (L, L^*) if and only if the pair (L, L^*) defines differential Gerstenhaber algebras [114]. This means that a differential operator d^* (d) is compatible with the Schouten-Nijenhuis bracket $[\cdot, \cdot]_{\mathrm{S}}$ ($[\cdot, \cdot]_{\mathrm{S}}^*$) in L (L^*). This will be explicitly seen in the DFT viewpoint in the next section. In particular, we will explicitly show that the exterior algebras of DFT defined on the Kaluza-Klein and winding spaces are incompatible with the derivation condition need for the Lie bialgebroid.

3.6 Doubled structures of other algebroids

Now, we are interested in what kind of algebroids equipped with the C-bracket are allowed other than the Vaisman and the Courant algebroids. This section discusses algebroid structures given by C-bracket more generally. First, we focus on each Axiom and write down all the possible algebroid structures that with the C-bracket. In general, Axioms C1-C5 of a Courant algebroid are not independent with each other [110]. Indeed, Axiom C5 implies C3, and C2 implies C4. Only Axiom C1 is independent of the other Axioms. Therefore, the possible combinations of Axioms in the general case are found to be

$$
\begin{aligned}
&(C1), \quad (C2, C4), \quad (C3), \quad (C4), \quad (C3, C5), \\
&(C1, C2, C4), \quad (C1, C3), \quad (C1, C4), \quad (C1, C3, C5), \\
&(C2, C3, C4), \quad (C2, C3, C4, C5), \\
&(C3, C4), \quad (C3, C4, C5) \\
&(C1, C2, C3, C4), \quad (C1, C2, C3, C4, C5), \\
&(C1, C3, C4), \quad (C1, C3, C4, C5)
\end{aligned} \tag{3.65}
$$

If we adopt the C-bracket in (3.65) to construct algebroids with doubled structure, then C3 and C5 are automatically satisfied. The possible combinations are limited to the following..

$$
\begin{aligned}
&(C1, C3, C5), \quad (C2, C3, C4, C5), \quad (C3, C5), \\
&(C3, C4, C5), \quad (C1, C2, C3, C4, C5), \quad (C1, C3, C4, C5)
\end{aligned} \tag{3.66}
$$

In (3.66), (C1, C2, C3, C4, C5) and (C3, C5) correspond to the Courant and the Vaisman algebroids, respectively. An algebroid defined by (C2, C3, C4, C5) is known to be the pre-Courant algebroid [115]. An algebroid by (C3, C4, C5) has been introduced in [116] and is called the ante-Courant algebroid. On the other hand, the other possibilities (C1, C3, C5), (C1, C3, C4, C5) have not been discussed in the literature. Since Axiom C1 means the (modified) Jacobi identity, we call (C1, C3, C5) the Jacobi Vaisman algebroid while (C1, C3, C4, C5) the Jacobi ante-Courant algebroid. Note that, this "Jacobi" is not related to the Jacobi structure proposed by Lichnerowicz which is a generalization of the Poisson structure. All of these algebroids are summarized in Figure 3.1. We collectively call these the DFT algebroids. They are organized into two parts. One is those with the Jacobi identity corresponding to the right flow in Figure 3.1. The other is the ones without the Jacobi identity in the left flow.

3.6.1 Doubled structures of Jacobi-Vaisman and Jacobi ante-Courant algebroids

We first consider the Jacobi Vaisman algebroid in the series of the right flow in Figure 3.1. The Jacobi Vaisman algebroid is obtained by imposing Axiom C1 to a Vaisman algebroid. We start from the Vaisman algebroid $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ made by a pair of Lie algebroids (E, E^*) and impose Axiom C1 to that. As we have seen before, Axiom C1 needs conditions $J_1 = J_2 = 0$. The condition $J_1 = 0$ is nothing but the derivation condition (3.18) itself. On

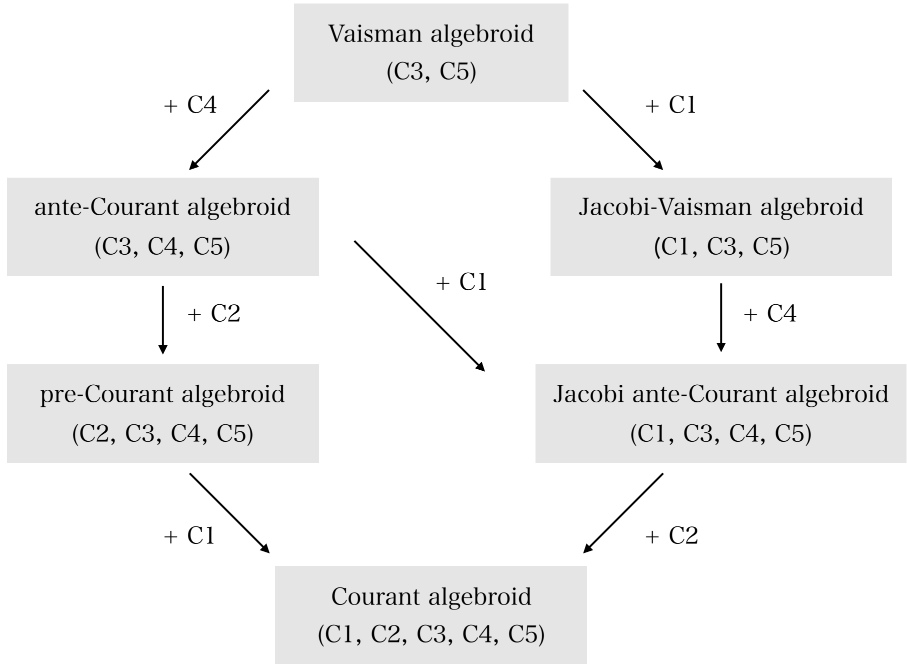

Figure 3.1: A list of algebroids allowed by the combinations of Axioms C1- C5. They are classified into the two sequences according to the logical structures of axioms.

the other hand, if we assume the derivation condition (3.18), we find

$$\begin{aligned} 0 &= (\mathcal{L}_{\mathrm{d}f}X + [\mathrm{d}_*f, X]_E) \wedge Y \\ &= -f\Big(\mathrm{d}_*[X,Y]_E + \mathcal{L}_Y\mathrm{d}_*X - \mathcal{L}_X\mathrm{d}_*Y\Big) + \Big(\mathrm{d}_*[X,fY]_E - \mathcal{L}_X\mathrm{d}_*(fY) + \mathcal{L}_{fY}\mathrm{d}_*X\Big) \end{aligned} \tag{3.67}$$

for any $X, Y \in \Gamma(E)$. A similar result holds for the dual of the (3.18). Therefore, we obtain the following relations

$$\mathcal{L}_{\mathrm{d}f}X = -[\mathrm{d}_*f, X]_L \tag{3.68}$$

If we consider $X = \mathrm{d}_*(e_1, e_2)_+, \xi = \mathrm{d}(e_1, e_2)_+$ in equation (3.68), we have $J_2 + \text{c.p.} = 0$. Therefore, it is sufficient to impose the derivation condition to make $J_1 + \text{c.p.} = 0$ and $J_2 + \text{c.p.} = 0$. Therefore, the derivation condition is the only necessary condition for the Jacobi Vaisman algebroid. However, as we will see in below, this condition induces Axioms C2 and C4. Let us first examine Axiom C4. Given the Vaisman algebroid $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$, we find the equation (3.22) is evaluated as (3.50). Note that the superscript $\bullet^*$ on the anchor means the adjoint operator, which is defined by the transposition of the original operator through the inner product $\langle\cdot,\cdot\rangle$. Namely, it is defined by $\langle \xi, \rho_E(X)\rangle = \langle \rho_{E^*}(\xi), X\rangle$ for any $X \in \Gamma(E), \xi \in \Gamma(E^*)$. The following is the summary of the anchor structures,

$$\begin{aligned} &\rho : E \to TM, \qquad \rho^* : T^*M \to E^*, \\ &\rho_* : E^* \to TM, \quad \rho_*^* : T^*M \to E. \end{aligned} \tag{3.69}$$

The rightmost side of the equation (3.50) seems to be generally non-zero. However, as we have clarified, the derivation condition (3.18) induces the condition (3.68). If we consider

$X = \mathrm{d}f$ in (3.68), we obtain

$$\mathrm{d}_*\Big(\rho_E\rho^*_{E^*}(\mathrm{d}_0 f)\cdot f\Big) = 0, \qquad {}^\forall f \in C^\infty(M). \tag{3.70}$$

This means

$$0 = \rho_E\rho^*_{E^*}(\mathrm{d}_0 f)\cdot f = \langle \rho_E\rho^*_{E^*}(\mathrm{d}_0 f), \mathrm{d}_0 f\rangle. \tag{3.71}$$

The result (A.65) is equivalent to the condition that the right-hand side of (3.50) vanishes. This also means the anti-symmetric property of the anchor map:

$$\rho_E\rho^*_{E^*} = -\rho_{E^*}\rho^*_E. \tag{3.72}$$

Therefore Axiom C4 is automatically satisfied by imposing the derivation condition (3.18).

Next, we clarify Axiom C2. Given the Vaisman algebroid $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$, we calculate the difference between the two sides of the equation (3.20). The result is

$$\begin{aligned}
&\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}})\cdot f - [\rho_{\mathrm{V}}(e_1), \rho_{\mathrm{V}}(e_2)]f \\
&= -\langle \xi_1, \big(\mathcal{L}_{\mathrm{d}f}X_2 - [X_2, \mathrm{d}_* f]_E\big)\rangle + \langle \xi_2, \big(\mathcal{L}_{\mathrm{d}f}X_1 - [X_1, \mathrm{d}_* f]_E\big)\rangle \\
&\quad + \frac{1}{2}\Big(\rho_E\rho^*_{E^*} + \rho_{E^*}\rho^*_E\Big)\mathrm{d}_0(\langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle)\cdot f.
\end{aligned} \tag{3.73}$$

It is obvious that the right-hand side of (3.73) vanishes by the conditions (3.68) and (3.72) that are induced by (3.18). Then, Axiom C2 is automatically satisfied due to the derivation condition.

In summary, as long as we employ the C-bracket and the doubled structure, it is impossible to construct the Jacobi Vaisman algebroid that satisfies only Axioms C1, C3, C5. The same is true even for the Jacobi ante-Courant algebroid. However, we stress that the result in this section does not mean that these algebroids are forbidden in general. Our discussion critically depends on the doubled structure of the C-bracket. If one does not persist on the doubled structure or the C-bracket, there is still room for these algebroids by defining a suitable bracket, instead of the C-bracket, that satisfies appropriate axioms.

3.6.2 Doubled structures of ante- and pre-Courant algebroids

Next, we consider the series of the left flow in Figure 3.1. We will clarify the compatibility conditions between the doubled structures and the C-bracket for these algebroids. Compared with the Jacobi Vaisman and the Jacobi ante-Courant algebroids, Axiom C1 is not required for ante- and pre-Courant algebroids.

First, we consider the ante-Courant algebroid. This is obtained by imposing Axiom C4 to a Vaisman algebroid. Again, we consider the Vaisman algebroid $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ made by a pair of Lie algebroids (E, E^*). The only condition that we need for ante-Courant algebroids is the anti-symmetric nature of the anchor (3.72). In the previous section, we showed that (3.72) is induced by the derivation condition but the converse is not true.

Therefore, even though the condition (3.72) is imposed, this does not imply the Jacobi identity and Axiom C1. The same is true for Axiom C2.

For the pre-Courant algebroid, we need to impose Axiom C2 in addition to C4 to the Vaisman algebroid. From the discussion in the previous section, the condition (3.68) for Axiom C2 implies the anti-symmetric nature of the anchor (3.72) required by Axiom C4. Then we conclude that only the condition for the pre-Courant algebroid is the equation (3.68).

A comment on the condition (3.68) is in order. When we take $X = \mathrm{d}_* g$ in the first equation in (3.68), we have

$$[\mathrm{d}_* g, \mathrm{d}_* f] = \mathcal{L}_{\mathrm{d}f}\mathrm{d}_* g = \mathrm{d}_*(\iota_{\mathrm{d}f}\mathrm{d}_* g), \quad f, g \in C^\infty(M). \tag{3.74}$$

Since $\iota_{\mathrm{d}f}\mathrm{d}_* g = \langle \mathrm{d}f, \mathrm{d}_* g\rangle = \langle \mathrm{d}_0 f, \rho_E \rho^*_{E^*}(\mathrm{d}_0 g)\rangle$, we find

$$[\mathrm{d}_* g, \mathrm{d}_* f] = \mathrm{d}_*\Big(\rho_E \rho^*_{E^*}(\mathrm{d}_0 g)[f]\Big). \tag{3.75}$$

Here we have used the notation $X[f] = \langle X, \mathrm{d}_0 f\rangle$ for vectors $X \in \Gamma(TM)$. Now we define a structure $\{g, f\}$ by

$$\{g, f\} = \pi(\mathrm{d}_* g)[f], \tag{3.76}$$

where $\pi = \rho_E \rho^*_{E^*}$. It is easy to show that this structure is skew-symmetric and possesses the bilinear nature. Since $\pi(\mathrm{d}_* g)$ belongs to $\Gamma(TM)$, the operator $\{g, \cdot\}$ acts on functions as a derivation. Furthermore, by acting ρ_E on the both sides of the relation (3.75), one can show that

$$\pi \mathrm{d}_0 \left(\Big\{\{g, f\}, h\Big\}\right) = \Big[[\pi \mathrm{d}_0 g, \pi \mathrm{d}_0 f], \pi \mathrm{d}_0 h\Big], \quad f, g, h \in C^\infty(M). \tag{3.77}$$

Since the right-hand side is given by the Lie bracket, the structure $\{\{g, f\}, h\}$ satisfies the Jacobi identity. These properties are enough to conclude that $\{g, f\}$ provides a non-trivial Poisson structure in M. One finds that $\bar{\pi} = \rho_{E^*}\rho^*_E$ also defines another Poisson structure. Although, this result was discussed first in [108] in the context of Lie bialgebroids, we stress that the essential property is (3.68) and the condition for the pre-Courant algebroid is necessary to define non-trivial Poisson structures in M.

3.7 From algebroids to algebras on group manifolds

In this section, we make a brief comment on the doubled structures discussed in this paper and those in group manifolds. The notion of the "double" has been originally proposed in the context of Hopf algebras by Drinfel'd [107]. A classical limit of this operation is implemented in Lie algebras [103, 104]. A well known fact is that a Lie algebra is defined by the left invariant vectors at the unit element of a group manifold. On the other hand, DFT on group manifolds has recently been considered [117]. In this setup, the manifest T-duality of DFT is generalized to the so-called Poisson-Lie T-duality [118]. An essential feature of the

Poisson-Lie T-duality lies in the structure of the Drinfel'd double of the underlying group manifold. Indeed, the Lie algebras associated with the abovementioned group is given by the Drinfel'd double. We note that it is possible to introduce the para-Hermitian nature even for group manifolds [119]. It is therefore natural to consider the relation between the doubled structure of the algebroids discussed here and the Lie algebras of the group manifold. The left invariant vector fields that define the Lie algebra of the group manifold are essentially given by a point on the group, namely, the unit element. Therefore, in order to find the associated Lie algebras from the algebroids, we consider only the unit element on the group manifold and restrict the vector space to the one for the left invariant vectors. This procedure is achieved by setting $\rho_{\mathrm{V}} = 0$. Then we have $\rho_E^* \mathrm{d}_0 = \mathrm{d} = 0, \rho_{E^*}^* \mathrm{d}_0 = \mathrm{d}_* = 0$, $\mathcal{D} = 0$. Under these conditions, Axioms C1 - C5 are rewritten as

Axiom C1 $\mathrm{Jac}(e_1, e_2, e_3) = 0$. The C-bracket becomes a Lie bracket.

Axiom C2 This becomes trivial by $\rho_{\mathrm{V}} = 0$.

Axiom C3 $[e_1, fe_2]_{\mathsf{C}} = f[e_1, e_2]_{\mathsf{C}}$ for any $f \in C^\infty(M)$. This shows the bilinearity of the Lie bracket.

Axiom C4 This becomes trivial by $\mathcal{D} = 0$.

Axiom C5 $([e_1, e_2]_{\mathsf{C}}, e_3)_+ + (e_1, [e_1, e_3]_{\mathsf{C}})_+ = 0$.

The last one is the compatibility condition between $[e_1, e_2]_{\mathsf{C}}$ and $(\cdot, \cdot)_+$. In general, a Lie algebra that has the compatible bilinear form $(\cdot, \cdot)_+$ is called a quadratic Lie algebra. It is also known that quadratic Lie algebras are infinitesimal algebras of Poisson-Lie groups. Therefore, we can see that a quadratic Lie algebra $(\mathcal{V}, [\cdot, \cdot]_{\mathsf{C}}, (\cdot, \cdot)_+)$ is obtained by the Courant algebroid made by the Drinfel'd double on the group manifold.

Now let us consider the Vaisman algebroid. As we have shown, the Jacobi identity is broken by the following quantities J_1, J_2:

$$
\begin{aligned}
J_1 &= \iota_{X_3}\Big(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1\Big) + \iota_{\xi_3}\Big(\mathrm{d}_*[X_1, X_2]_E - \mathcal{L}_{X_1}\mathrm{d}_*X_2 + \mathcal{L}_{X_2}\mathrm{d}_*X_1\Big), \\
J_2 &= \Big(\mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}\xi_3 + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*}\Big) - \Big(\mathcal{L}_{\mathrm{d}(e_1,e_2)_-}X_3 + [\mathrm{d}_*(e_1, e_2)_-, X_3]_E\Big).
\end{aligned}
\tag{3.78}
$$

When we set $\rho_{\mathrm{V}} = 0, \mathrm{d} = 0, \mathrm{d}_* = 0$, then it is obvious that $J_1 = J_2 = 0$. Thus, even if we consider the algebroid where Axiom C1 does not hold, going to the algebra, we end up with $\mathrm{Jac}(e_1, e_2, e_3) = 0$ and the bracket becomes a Lie bracket. Therefore, all the algebroids with doubled structure discussed in this paper reduce to the quadratic Lie algebras at the unit of the group manifold. Since the doubled structure of algebroids is essentially irrelevant to the group action, the DFT algebroids discussed in this paper are non-group generalizations of the quadratic Lie algebras defined by the Drinfel'd double. This would be a key property for the further generalizations of the Poisson-Lie T-duality.

4 Generalized and Doubled Geometry

In Chaper 3, the gauge symmetry of the DFT and related algebroid structures were discussed. Of particular importance is the fact that the Vaisman algebroid specified by the C-bracket was found to have a doubled structure. We would like to investigate how this doubled structure is expressed in the DFT. To do so, we should further understand the geometry of the DFT.

In general, as a geometrical framework related to T-duality is generalized geometry. This framework is presented by Hitchin [37]. It introduced the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, which was a generalized tangent bundle. It is a geometry with $O(D, D)$ symmetry, which is a T-duality group as a structure group. Generalised geometry has led to various physical results [43–45, 48, 120, 121].

On the other hand, as mentioned in Chapter 2, DFT was introduced In order to make T-duality explicit. DFT is established on the doubled spacetime $(x, \tilde{x})$ which is combination of the usual physical coordinates x and the winding coordinates $\tilde{x}$.This cannot be represented by the known Riemannian geometry, nor by generalized geometry. This suggests the existence of a new geometry in the background. This geometry is called doubled geometry. The mathematical understanding of doubled spacetime has progressed rapidly in recent years. More recently, in relation to para-Hermitian geometry [72–76] and Born geometry [76]. They are expected to serve as a framework for "grobal" doubled geometry.

In this Chapter, we first describe the basic setting of generalised geometry and clarify its relation to Courant algebroid discussed in Chapter 2. Furthermore, we construct para-Hermitian geometry and Born geometry and discuss their relation to generalised geometry.

4.1 Generalized geometry

The main idea of generalised geometry is, roughly speaking replace the tangent bundle TM by a generalised tangent bundle constructed by the direct sum $TM \oplus T^*M$, and on which differential geometry is to be developed.

Definition 4.1.1. The generalised tangent bundle $\mathbb{T}M$ of a manifold M is the direct sum of the ordinary tangent bundle TM and the cotangent bundle T^*M $TM \oplus T^*M$.

From this definition, it is clear that $TM \oplus T^*M$ is a vector space of dimension $2D$. However, there is no restriction on the basespace (manifold) M. In this case, the inner product in $TM \oplus T^*M$ is naturally given as follows.

$$\langle X_1 + v_1 + X_2 + v_2 \rangle = \frac{1}{2}(v_1(X_2) + v_1(X_2)) \tag{4.1}$$

The following forms of metrology are therefore also naturally introduced.

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.2}$$

This η_{MN} is the $O(D,D)$-invariant metric. The structure group of $\mathbb{T}M$ is thus an $O(D,D)$ group. To preserve the invariance of the inner product, the generator $\mathbb{G}$ satisfies the following equation.

$$\langle \mathbb{G}(X_1 + v_1), X_2 + v_2 \rangle + \langle X_1 + v_1, \mathbb{G}(X_2 + v_2) \rangle = 0. \tag{4.3}$$

The generator $\mathbb{G}$ of the structure group $O(D,D)$ is determined as follows.

$$\mathbb{G} = \begin{pmatrix} A & \beta \\ B & A^{-1t} \end{pmatrix} \tag{4.4}$$

However, $A \in \mathrm{End}(TM)$ and B is the antisymmetric 2-form $\Gamma(\wedge^2 T^*M)$. Also, β is the antisymmetric 2-vector $\Gamma(\wedge^2 TM)$. This is a composition of three transformations: the general linear transformation $G(D)$ with A, the B transformation and the β transformation. In particular, the B-transform is defined as a shift with respect to the vector field X as follows.

$$e^B : (X_1 + v_1) \to (X_1 + v_1) + B(X). \tag{4.5}$$

This can be written in matrix form as follows.

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in O(D,D). \tag{4.6}$$

Physically, this 2-form B corresponds to the Kalb-ramond field $B_{\mu\nu}$. The generalized Lie derivative $\mathcal{L}$ generated by $X_1 + B \in (TM \oplus \wedge^2 T^*M)$ can then be written as follows [122].

$$\mathcal{L}_{X_1+B}(X_2 + v_2) = \mathcal{L}_{X_1}(X_2 + v_2) - \iota_{X_2} B. \tag{4.7}$$

Here, it is the usual Lie derivative with $\mathcal{L}_{X_1}$ vector field $X \in \Gamma(TM)$. In this case, the first entry on the right-hand side is derived from the diffeomorphism map and the second from the B-transform. This is the reason why it is explained that "In generalized geometry, the diffeomorphism by the metric $g_{\mu\nu}$ and the gauge transformation in the anti-symmetric 2-form field $B_{\mu\nu}$ are integrated and coexist". However, in essence, if the structure group is $O(D,D)$, $g_{\mu\nu}$ and $B_{\mu\nu}$ can coexist, not only in generalised geometry. The doubled geometry described in next section is a good example. The generalised Lie derivative $\hat{L}$ of the DFT described in Chapter 3 also includes both the differential homomorphic mapping by the metric $g_{\mu\nu}$ and the gauge transformation of the $B_{\mu\nu}$ field.

Also, the generaliezd Lie derivative $\mathcal{L}$ generated from the vector field $X + v$ given by $\Gamma(\mathbb{T}M)$ is [122], which can be written as follows.

$$\begin{aligned}\mathcal{L}_{X_1+v_1}(X_2+v_2) &= \mathcal{L}_{X_1}X_2 + \mathcal{L}_{X_1}v_2 - \iota_{X_2}\mathrm{d}v_1 \\ &= [X_1, X_2] + \mathcal{L}_{X_1}v_2 - \iota_{X_2}\mathrm{d}v_1 \\ &\equiv (X_1+v_1)\circ(X_2+v_2)\end{aligned} \tag{4.8}$$

Here, $[X_1, X_2]$ is a Lie bracket (or commutator) for an ordinary vector field in $\Gamma(TM)$, and The ι, d are the inner and exterior products, respectively (see Sec 3.1). The right-hand side of (4.8) is exactly the dorfman bracket operator $\circ$ of Sec 3.2. Therefore, the structure generated by the generaliezd Lie derivative $\mathcal{L}$ in generalised geometry is a Courant algebroid. In general, the antisymmetric construction of the dorfman brackt operator is a Courant bracket. From (4.8), we can now give a concrete example of Courant bracket $[\cdot,\cdot]_c$ as follows.

$$\begin{aligned}[X_1+v_1, X_2+v_2]_c &= \frac{1}{2}\big((X_1+v_1)\circ(X_2+v_2) - (X_2+v_2)\circ(X_1+v_1)\big) \\ &= \frac{1}{2}\big(([X_1,X_2] + \mathcal{L}_{X_1}v_2 - \iota_{X_2}\mathrm{d}v_1) - ([X_2,X_1] + \mathcal{L}_{X_2}v_1 - \iota_{X_1}\mathrm{d}v_2)\big) \\ &= \frac{1}{2}\big(([X_1,X_2] + \mathcal{L}_{X_1}v_2 - \mathcal{L}_{X_2}v_1 + \mathrm{d}\iota_{X_2}v_1) \\ &\qquad - ([X_2,X_1] + \mathcal{L}_{X_2}v_1 - \mathcal{L}_{X_1}v_2 + \mathrm{d}\iota_{X_1}v_2)\big) \\ &= [X_1,X_2] + \mathcal{L}_{X_1}v_2 - \mathcal{L}_{X_2}v_1 - \frac{1}{2}\mathrm{d}(\iota_{X_1}v_2 - \iota_{X_2}v_1)\end{aligned} \tag{4.9}$$

This Courant bracket is also the earliest historically constructed concrete example of [109]. Moreover, for this Courant bracket, a twist by 3-form $H \in \Gamma(\wedge^3 T^*M)$ such that $\mathrm{d}H = 0$ also The structure of the Courant algebroid as a whole is preserved.

$$[X_1+v_1, X_2+v_2]_H = [X_1+v_1, X_2+v_2]_c + \iota_{X_2}\iota_{X_1}H \tag{4.10}$$

This H physically corresponds to H-flux. The twisted algebroid is discuss in Sec 5.6. The generalised geometry is revolutionary in that it naturally incorporates B-fields and H-fluxes. Physically, it gives a geometrical picture for the NS-NS sector of the supergravity theory derived from string theory. This framework was applied to the study of (non-)geometric flux in SUGRA [43–45, 120, 121]. Its relationship with the non-linear sigma model has also been studied. The non-linear sigma model is a scalar field theory with non-linear coupling that describes string theory in a curved background. In its Hamiltonian form, the diffeomorphism by the metric $g_{\mu\nu}$ and the gauge transformation of the Kalb-Ramond field $B_{\mu\nu}$ can be regarded not as a mere field transformation but as a (part of) canonical transformation on a phase space. The canonical transformation corresponds to a general coordinate transformation on the phase space and can be regarded as a geometric symmetry of the phase space. Indeed, the generators generating the diffeomorphism and the gauge transformation of the B-field in the Hamiltonian form can be regarded as the basic geometric structures of generalised geometry (generalised tangent bundle and algebroid as gauge algebra). It is known that These are one-to-one correspondence [46]. Gates-Hull-Rocek geometry discovered in the study of $2D$ non-linear sigma models with $\mathcal{N} = (2,2)$ SUSY [47] (also called bi-Hermitian geometry) and the generalised Kähler geometry defined in the framework of generalised geometry are also known to be equivalent [38, 48]. Here, we only introduce the definition of the generalised Kähler structure [38]. A generalised Kähler structure is defined using a generalised (almost) complex structure. First, the generalised (almost) complex structure is the following structure.

Definition 4.1.2. A generalized almost complex structure is an endomorphism $J : TM \oplus T^*M \to TM \oplus T^*M$ for $\mathbb{T}M = TM \oplus T^*M$. J needs to satisfy $\langle J\cdot, J\cdot\rangle = \langle\cdot,\cdot\rangle$and $J^2 = -1$. From the properties of J, the vector space defined by J is of even-dimension. Furthermore, if $\Gamma(\mathbb{T}M)$ is closed in Courant bracket $[\cdot,\cdot]_c$ (4.9), J is called the generalised complex structure..

Using J, the generalised Kähler structure is then defined as follows.

Definition 4.1.3. A generalized Kähler structure is a pair of two generalized complex structures $(J_1, J_{2)}$ such that In particular, one in which $J_1J_2 - J_2J_1 = 0$ and the product J_1J_2 gives a positive-definite metric on $\mathbb{T}M$.

Thus, there is no doubt that generalised geometry is a very useful framework for considering T-duality and related physics. However, it should be noted that only tangent bundles are generalised. It is (type II) supergravity that is more compatible with generalised geometry, not DFT. DFT geometry is a framwork such that the basespace M itself is doubled.

4.2 Doubled geometry

Doubled geometry was proposed by Hull in 2004 [54]. In doubled geometry, the bottom space itself is doubled to $2D$-dimensions. It is also a different framework from the generalized geometry described above in that it doubles the bottom space itself to $2D$-dimensions. Doubled geometry is known to be highly compatible with the doubled spacetime $(\tilde{x}, x)$ of DFT.

Of course, the geometrical aspects of doubled spacetime were not always fully developed. Immediately after doubled geometry started to attract attention as a geometry for DFT, the following geometry was developed Many arguments relying on the space-time coordinate $(\tilde{x}, x)$, [123, 124] and the frame-like formulation [51, 52] and the metric-like formulation [124–126]. The [127] attempted to bring these together and understand them more universally as an analogy of Riemannian geometry. It was proposed that this structure is naturally incorporated in a para-Hermitian (Kähler) manifold [72, 73]. The para-Hermitian structure is a basic ingredient to understand the doubled nature of space-time behind DFT. In the following, we exhibit basic materials related to para-Hermitian geometries [74, 76] and then discuss algebroid structures realized in DFT. We introduce this geometry in the next section.

4.2.1 para-Hermitian geometry

Before discussing the para-Hermitian structure, we first define an almost para-complex manifold.

Definition 4.2.1. An almost para-complex manifold $(\mathcal{M}, K)$ is a differential manifold $\mathcal{M}$ with a vector bundle endomorphism $K : T\mathcal{M} \to T\mathcal{M}$ where $K^2 = +1$. This K is called the almost para-complex structure that satisfies the condition dim $\ker(K+1) =$ dim $\ker(K-1)$.

Obviously, the almost para-complex structure is a real analogue of the almost complex structure $J^2 = -1$. Given an almost para-complex structure K, the tangent bundle $T\mathcal{M}$ is decomposed into the eigenbundles $L, \tilde{L}$ associated with the eigenvalues $K = \pm 1$. This decomposition is performed via the projection operators $P, \tilde{P}$ that map elements in $T\mathcal{M}$ to those in L or $\tilde{L}$:

$$P = \frac{1}{2}(1+K), \qquad \tilde{P} = \frac{1}{2}(1-K). \tag{4.11}$$

The subbundles $L, \tilde{L}$ are distributions of $T\mathcal{M}$. We stress that the para-complex structure K provides a natural decomposition of vectors in doubled space-time.

Definition 4.2.2. A distribution is defined as follows. Let M be an m-dimensional C^∞-manifold. For any $x \in M$, we can consider an n-dimensional $(n \leq m)$ subbundle $\Delta_x \subset T_xM$. We then consider a neighborhood of x, $N_x \subset M$. In N_x, there are n independent vector fields $X_1, \ldots, X_n$. They define a linear span for any point $y \in N_x$. Namely, these n vector fields generate a subbundle $\Delta_y = \{X_1(y), \ldots, X_n(y)\}$. For any $x \in M$, with a set Δ_x, we call $\Delta = \bigcup_{x\in M} \Delta_x$ the n-dimensional distribution over M. This is also known as the C^∞ n-plane distribution over M. A set of the smooth vector fields $\{X_1, \ldots, X_n\}$ is called the local basis of Δ.

We now discuss the notion of integrability. The integrability of a distribution is properly represented by the Frobenius theorem. The Frobenius theorem is understood as a property of vector fields. For any vector fields $X, Y \in \Gamma(L)$ where L is a distribution, if their Lie bracket $[X, Y]_L$ belongs to L, then the distribution L is called involutive. The Frobenius theorem states that a distribution L (resp. $\tilde{L}$) is completely integrable if and only if L (resp. $\tilde{L}$) is involutive. When the eigenbundle L (resp. $\tilde{L}$) defined on an almost para-Hermitian manifold is involutive, then the tensors $N_P, N_{\tilde{P}}$ defined in the following vanish:

$$N_P(X,Y) = \tilde{P}[P(X), P(Y)], \qquad N_{\tilde{P}}(X,Y) = P[\tilde{P}(X), \tilde{P}(Y)], \tag{4.12}$$

where $X, Y \in \Gamma(T\mathcal{M})$. We can define the Nijenhuis tensor associated with K by adding the two tensors in (4.12): $N_K(X,Y) = N_P(X,Y) + N_{\tilde{P}}(X,Y)$. This is again a real analogue of the Nijenhuis tensor defined on an ordinary complex manifold:

$$N_K(X,Y) = \frac{1}{4}\big\{[K(X), K(Y)] + [X,Y] - K\big([K(X), Y] + [X, K(Y)]\big)\big\}. \tag{4.13}$$

The Nijenhuis tensor is a torsion on a (para-)complex manifold. When N_K vanishes, K is integrable. Then the definition of a para-complex manifold is given as follows:

	$\mathrm{d}\omega \neq 0$	$\mathrm{d}\omega = 0$
$N_K \neq 0$	almost para-Hermitian (almost symplectic)	almost para-Kähler (symplectic)
$N_K = 0$	para-Hermitian (almost symplectic)	para-Kähler (symplectic)

Table 4.1: The integrability and closeness of ω.

Definition 4.2.3. When K is integrable, namely, the Nijenhuis tensor N_K vanishes identically, then an almost para-complex manifold $(\mathcal{M}, K)$ is a para-complex manifold.

Contrast to the ordinary complex manifolds, the notion of integrability for the two distributions L and $\tilde{L}$ are totally independent with each other. Namely, the integrability of L is defined through the condition $N_P(X,Y) = 0$ for any $X, Y \in \Gamma(T\mathcal{M})$. This does not imply $N_{\tilde{P}} = 0$ in general. Since the integrability condition is independent for L and $\tilde{L}$, we can define a half-integrability in a para-complex manifold [74, 76]:

Definition 4.2.4. An L-para-complex manifold is an almost para-complex manifold $(\mathcal{M}, K)$ where only L is integrable. The same is true for $\tilde{L}$. When the L-para-complex and the $\tilde{L}$-para-complex conditions are satisfied simultaneously, then $(\mathcal{M}, K)$ is a para-complex manifold.

We next define an almost para-Hermitian manifold by introducing a metric η,

Definition 4.2.5. An almost para-Hermitian manifold $(\mathcal{M}, \eta, K)$ is an almost para-complex manifold $\mathcal{M}$ equipped with a neutral metric $\eta : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$ which satisfies the compatibility condition $\eta(K\cdot, K\cdot) = -\eta(\cdot,\cdot)$. η is called the para-Hermitian metric.

By its definition, the distribution L is maximally isotropic with respect to η. Namely, for any $X, Y \in \Gamma(L)$, since they are elements of the eigenbundle with $K = 1$, we have $\eta(X,Y) = 0$ for a para-Hermitian metric η. The same is true even for $\tilde{L}$. Since η is neutral, it follows that L and $\tilde{L}$ have the same rank $D = \frac{1}{2}\dim \mathcal{M}$. Given an almost para-complex structure K and a compatible metric η, then we can define a non-degenerate 2-form $\omega = \eta K$. This can be seen as an almost symplectic structure on $\mathcal{M}$ and it is not closed in general $\mathrm{d}\omega \neq 0$. This means that an almost para-Hermitian manifold $(\mathcal{M}, K, \eta)$ is an almost symplectic manifold $(\mathcal{M}, \omega)$ and vice-versa. When ω is closed, $(\mathcal{M}, K, \eta)$ and $(\mathcal{M}, \omega)$ are said to be almost para-Kähler and symplectic, respectively (see Table 4.1.) We note that a symplectic manifold is a Poisson manifold.

The compatibility between η and ω results in that L and $\tilde{L}$ are Lagrangian subbundles with respect to ω. Namely, for any $X, Y \in \Gamma(L)$ (resp. $\Gamma(\tilde{L})$), we have $\omega(X,Y) = 0$. We note that even for the case where ω is not closed, we can define a Lagrangian subspace of ω. Given the almost structures, an analogue of a Hermitian manifold is defined:

Definition 4.2.6. When $(\mathcal{M}, K)$ is an L-para-complex manifold, then an almost para-Hermitian manifold $(\mathcal{M}, \eta, K)$ is an L-para-Hermitian manifold. This is also the same

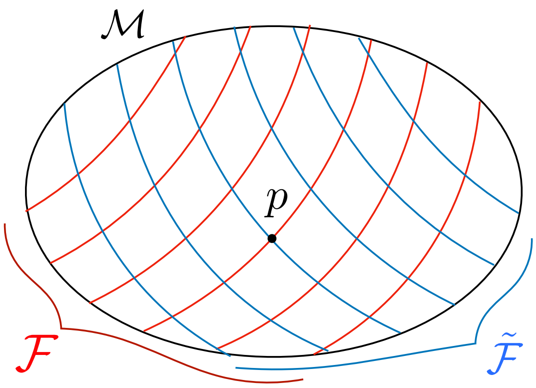

Figure 4.1: Image of para-Hermitian manifold and foliation $\mathcal{F}, \tilde{\mathcal{F}}$.

for $\tilde{L}$. An almost para-Hermitian manifold that satisfies both the L- and $\tilde{L}$-integrability conditions is a para-Hermitian manifold.

The subbundles $L, \tilde{L}$ on a para-Hermitian manifold is therefore Dirac structures. Namely, they are maximally isotropic with respect to η and involutive.

An alternative representation of the Frobenius theorem states that a subbundle $E \subset T\mathcal{M}$ is integrable if and only if it is defined by a regular foliation of $\mathcal{M}$. Namely, an integrable subbundle $E \subset T\mathcal{M}$ defines the tangent bundle of a foliation $\mathcal{F}$ in $\mathcal{M}$. Therefore when L and $\tilde{L}$ are integrable, then they have foliation structures:

$$L = T\mathcal{F} \qquad \text{and} \qquad \tilde{L} = T\tilde{\mathcal{F}}. \tag{4.14}$$

Here the foliation $\mathcal{F}$ (resp. $\tilde{\mathcal{F}}$) is given by the union of leaves $\coprod_{[p]} M_{[p]}$. A leaf M_p is a subspace of $\mathcal{F}$ (resp. $\tilde{\mathcal{F}}$) that pass through a point $p \in \mathcal{M}$ and its tangent vectors are specified by L (resp. $\tilde{L}$). The index space in the union is the leaf space $\mathcal{M}/\mathcal{F}$ or $\mathcal{M}/\tilde{\mathcal{F}}$. For $\mathcal{F}$, the local coordinate x^μ is given along a leaf M_p while the one for the transverse directions to leaves is $\tilde{x}_\mu$. This means that $\tilde{x}_\mu$ is a constant on a leaf M_p in $\mathcal{F}$ (fig. 4.1).

The metric η over $\mathcal{M}$ can be seen as a map $\eta : T\mathcal{M} = L \oplus \tilde{L} \to T^*\mathcal{M} = L^* \oplus \tilde{L}^*$. Then the metric η defines the following two isomorphisms:

$$\phi^+ : \tilde{L} \to L^* \qquad \text{and} \qquad \phi^- : L \to \tilde{L}^*. \tag{4.15}$$

They map vectors in $\tilde{L}$ (resp. L) to forms in L^* (resp. $\tilde{L}^*$). The converse is also true. Given these isomorphisms, the following new isomorphisms are naturally defined:

$$\Phi^+ : T\mathcal{M} \to L \oplus L^* \qquad \text{and} \qquad \Phi^- : T\mathcal{M} \to \tilde{L} \oplus \tilde{L}^*. \tag{4.16}$$

In particular, the map Φ^+ is utilized to relate DFT and generalized geometry and it is called the natural isomorphism.

4.2.2 Born geometry

So far, we discuss that para-Hermitian geometry is based on the para-complex geometry $(\mathcal{M}, K)$ with the newtral metric η. Furthermore, as a next step, by adding another metric

$\mathcal{H}$ with the signature $(2D, 0)$, para-Hermitian geometry can be further generalised. Such a geometry is called Born geometry [76]. Discussions on sigma models based on Born geometry are given in [77, 78, 119]. The relation between Born geometry and generalised geometry (generalised Kähler structure) is discussed in [79].

Definition 4.2.7. Let $(\mathcal{M}, \eta, \omega)$ be a para-Hermitian manifold. Let $\mathcal{H}$ be a Riemannian metric. The following two equations are introduced as satisfying the following two equations.

$$\eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta, \quad \omega^{-1}\mathcal{H} = \mathcal{H}^{-1}\omega. \tag{4.17}$$

In this case, the pair of structures $(\eta, \omega, \mathcal{H})$ is called the Born structure on $mathcal M$. Also,$(\mathcal{M}, \eta, \omega, \mathcal{H})$ is called Born geometry.

The Born structure $(\eta, \omega, \mathcal{H})$ contains three structures from the combination of each element. The first is the para-Hermitian structure (ω, K) mentioned in the previous section, and the following combatibility exists for K.

$$K^2 = 1, \quad \omega(KX, KY) = -\omega(X, Y). \tag{4.18}$$

The second is the combination of $(\eta, \mathcal{H})$. This structure is called the chiral structure J. $J = \eta^{-1}\mathcal{H}$, and the following compatibility exists for J.

$$J^2 = 1, \quad \eta(JX, JY) = \eta(X, Y). \tag{4.19}$$

The third is a combination of $(\omega, \mathcal{H})$. This structure is called the almost Hermitian structure I. $I = \mathcal{H}^{-1}\omega$ and has the following compatibility.

$$I^2 = -1, \quad \mathcal{H}(IX, IY) = \mathcal{H}(X, Y). \tag{4.20}$$

These three structures form a para-quotanic structure as follows. where , is an anti-commutator

$$-I^2 = J^2 = K^2 = 1, \quad \{I, J\} = \{J, K\} = \{K, I\} = 0, \quad KJI = 1. \tag{4.21}$$

Finally, from studies on geometric quantisation and the geometry of quantum mechanics, it is It is well known that the geometrical structure underlying the quantisation process is a Hermitian structure. In this context, Born structure can be understood as a phase space (of relativistic particles) and The symplectic structure defines a Hamiltonian flow on the phase space. The relationship between each structure can be organised as in Table 4.2.

Using the $2D$-Riemannian metric $\mathcal{H}$, the Born geometry version of Levi-Civita connection as an analogy for Riemannian geometry in [76]. Torsion and Riemann curvature are also calculated. In particular, the Riemannian curvature in Born geometry reproduces the Riemann curvature (or generalized Riemannian tensor in DFT) initially presented in [127].

$-I^2 = J^2 = K^2 = 1$	$I, J = J, K = K, I = 0$	$KJI = 1$
$I = -\omega^{-1}\mathcal{H} = \mathcal{H}^{-1}\omega$	$J = \eta^{-1}\mathcal{H} = \mathcal{H}^{-1}\eta$	$K = \eta^{-1}\omega = \omega^{-1}\eta$
$\mathcal{H}(IX, IY) = \mathcal{H}(X, Y)$	$\eta(IX, IY) = -\eta(X, Y)$	$\omega(IX, IY) = \omega(X, Y)$
$\mathcal{H}(JX, JY) = \mathcal{H}(X, Y)$	$\eta(JX, JY) = \eta(X, Y)$	$\omega(JX, JY) = -\omega(X, Y)$
$\mathcal{H}(KX, KY) = \mathcal{H}(X, Y)$	$\eta(KX, KY) = -\eta(X, Y)$	$\omega(KX, KY) = -\omega(X, Y)$

Table 4.2: The relationship between I, J, K and compatibility.

5 Geometric Realization of Algebroid

In Chaper 3, we discuss the algebroids and the doubled structures. It is shown that the Vaisman algebroid defined by the C-bracket has a doubled structure. In Chapter 4, we discuss the para-Hermitian geometry and its generalisation, the Born geometry, as the geometry of DFT. We confirmed that doubled spacetime $(\tilde{x}, x)$ can be naturally expressed under these geometries.

In this chapter, we reproduce the Vaisman algebroid the DFT by C-bracket on the para-Hermitian geometry. First, some necessary structures for component computation are introduced. We then reproduce the Lie algebroids on para-Hermitian manifold. Using these results, we discuss the algebraic origin of strong constraint.

5.1 Para-Dolbeault cohomology

In the previous section, it was confirmed that the para-Hermitian manifold $\mathcal{M}$ is useful as a mathematical description of doubled spacetime. However, to actually construct a Vaisman algebroid on $\mathcal{M}$, it is necessary to define operators such as external derivatives and inner products, and to write them down so that they can be used to compute the components.

In this subsection, we define a para-Dolbeault cohomology in $L, \tilde{L}$. It is always true that there is a natural exterior algebra on the tangent bundle over an almost para-complex manifold $\mathcal{M}$. We introduce the section of $\wedge^k T\mathcal{M}$ (the totally anti-symmetric k-th tensor products of $T\mathcal{M}$) and denote it as $\hat{\mathcal{A}}^k(\mathcal{M})$. Since $L, \tilde{L}$ are subbundles in $T\mathcal{M}$, we can define exterior algebras in $\Gamma(L)$ and $\Gamma(\tilde{L})$. If we define $\mathcal{A}^{r,s}(\mathcal{M})$ as the section of $(\wedge^r L) \wedge (\wedge^s \tilde{L})$, then, we obtain the following decomposition:

$$\hat{\mathcal{A}}^k(\mathcal{M}) = \bigoplus_{k=r+s} \mathcal{A}^{r,s}(\mathcal{M}). \tag{5.1}$$

Here we have defined the canonical projection operator $\pi^{r,s} : \hat{\mathcal{A}}^{r+s}(\mathcal{M}) \to \mathcal{A}^{r,s}(\mathcal{M})$ that is induced by P and $\tilde{P}$ (see the explicit example in the next subsection). We then define the exterior derivatives acting on L and $\tilde{L}$:

$$\tilde{\mathrm{d}} : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r+1,s}(\mathcal{M}), \tag{5.2}$$

$$\mathrm{d} : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r,s+1}(\mathcal{M}). \tag{5.3}$$

They are called the para-Dolbeault operators and have the following properties:

$$\mathrm{d}^2 = 0, \qquad \tilde{\mathrm{d}}^2 = 0, \qquad \mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d} = 0. \tag{5.4}$$

Due to the nilpotency of the para-Dolbeault operators, we can define the para-Dolbeault cohomology. This is a real analogue of the Dolbeault cohomology defined in a complex manifold. For any $A \in \Gamma(L)$, $\alpha \in \Gamma(\tilde{L})$, the interior products ι_A, $\tilde{\iota}_\alpha$ are defined:

$$\iota_A : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r-1,s}(\mathcal{M}) \qquad \text{and} \qquad \tilde{\iota}_\alpha : \mathcal{A}^{r,s}(\mathcal{M}) \to \mathcal{A}^{r,s-1}(\mathcal{M}). \tag{5.5}$$

By these operations, we define the Lie derivatives on L and $\tilde{L}$:

$$\mathcal{L}_A \xi = (\mathrm{d}\iota_A + \iota_A \mathrm{d})\xi, \qquad \tilde{\mathcal{L}}_\alpha \xi = (\tilde{\mathrm{d}}\tilde{\iota}_\alpha + \tilde{\iota}_\alpha \tilde{\mathrm{d}})\xi. \tag{5.6}$$

Here $A \in \Gamma(L)$, $\alpha \in \Gamma(\tilde{L})$, $\xi \in \mathcal{A}^{r,s}(\mathcal{M})$ are arbitrary (multi-)vectors. By a para-Hermitian metric η, there is a natural $C^\infty(\mathcal{M}, \mathbb{R})$-bilinear map on $\mathcal{A}^{1,0}(\mathcal{M}) \times \mathcal{A}^{0,1}(\mathcal{M})$. We call this the (symmetric) pairing. The pairing is denoted as $(\alpha, A) \mapsto \langle\!\langle \alpha, A \rangle\!\rangle$. This is an analogue of the inner products between vectors and forms on $T\mathcal{M}$ and $T^*\mathcal{M}$. Here we note that L and $\tilde{L}$ are not necessarily dual with each other. When $A \in \mathcal{A}^{r,0}(\mathcal{M})$, $\alpha \in \mathcal{A}^{0,s}(\mathcal{M})$, $r \neq s$, then the pairing is given by $\langle\!\langle \alpha, A \rangle\!\rangle = 0$. In particular, for $\alpha \in \mathcal{A}^{0,s}(\mathcal{M})$ and $A_1, \dots, A_s \in \mathcal{A}^{1,0}(\mathcal{M})$, we write

$$\langle\!\langle \alpha, A_1 \wedge \cdots \wedge A_s \rangle\!\rangle = \alpha(A_1, \dots, A_s). \tag{5.7}$$

Similarly, for $A \in \mathcal{A}^{r,0}(\mathcal{M})$ and $\alpha_1, \dots, \alpha_r \in \mathcal{A}^{0,1}(\mathcal{M})$ we write

$$\langle\!\langle \alpha_1 \wedge \cdots \wedge \alpha_r, A \rangle\!\rangle = A(\alpha_1, \dots, \alpha_r). \tag{5.8}$$

Now we express the interior products (5.5) by these quantities. For $\alpha \in \mathcal{A}^{0,s}(\mathcal{M})$, $\iota_A \alpha$ is an element of $\mathcal{A}^{0,s-1}(\mathcal{M})$. Therefore, for $A_1, \dots, A_{s-1} \in \mathcal{A}^{1,0}(\mathcal{M})$, it is written as

$$\iota_A \alpha(A_1, \dots, A_{s-1}) = \alpha(A, A_1, \dots, A_{s-1}). \tag{5.9}$$

Similarly, for $A \in \mathcal{A}^{r,0}(\mathcal{M})$, $\tilde{\iota}_\alpha A$ is an element of $\mathcal{A}^{r-1,0}(\mathcal{M})$. Therefore by $\alpha_1, \dots, \alpha_{r-1} \in \mathcal{A}^{0,1}(\mathcal{M})$, it is written as

$$\tilde{\iota}_\alpha A(\alpha_1, \dots, \alpha_{r-1}) = A(\alpha, \alpha_1, \dots, \alpha_{r-1}). \tag{5.10}$$

The interior product ι_A (resp. $\tilde{\iota}_\alpha$) is a degree -1 derivation on the exterior algebras of $\tilde{L}$ (resp. L):

$$\begin{aligned} \iota_A(\alpha \wedge \beta) &= (\iota_A \alpha) \wedge \beta + (-1)^s \alpha \wedge \iota_A \beta, \\ \tilde{\iota}_\alpha(A \wedge B) &= (\tilde{\iota}_\alpha A) \wedge B + (-1)^r A \wedge \tilde{\iota}_\alpha B. \end{aligned} \tag{5.11}$$

Here $\alpha \in \mathcal{A}^{0,s}(\mathcal{M})$, $\beta \in \mathcal{A}^{0,\bullet}(\mathcal{M})$, $A \in \mathcal{A}^{r,0}(\mathcal{M})$, $B \in \mathcal{A}^{\bullet,0}(\mathcal{M})$.

5.2 Vaisman algebroid on para-Hermitian manifold

Now we discuss the algebroid structure governed by the C-bracket (2.23) in DFT. The doubled space-time on which DFT is defined is given by a flat para-Hermitian manifold

$\mathcal{M}$ whose local coordinate is x^M [74]. The tangent space $T\mathcal{M}$ is spanned by ∂_M ($M = 1, \ldots, 2D$). Vector fields on $T\mathcal{M}$ are decomposed by the projection operators $P, \tilde{P}$ defined by the para-complex structure K. Namely, for $\Xi = \Xi^M \partial_M \in T\mathcal{M}$, we have

$$\Xi^M \partial_M = A^\mu(x, \tilde{x})\partial_\mu + \alpha_\mu(x, \tilde{x})\tilde{\partial}^\mu, \tag{5.12}$$

where $A \in \Gamma(L)$, $\alpha \in \Gamma(\tilde{L})$. Here $x^M = (x^\mu, \tilde{x}_\mu)$ is the induced decomposition of the local coordinate on the base space $\mathcal{M}$. Therefore L is spanned by ∂_μ ($\mu = 1, \ldots, D$) while $\tilde{L}$ is spanned by $\tilde{\partial}^\mu$ in the DFT framework. In a flat para-Hermitian manifold, there is always a local frame where the para-Hermitian metric η is expressed as

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.13}$$

Since this metric induces a map $L \to \tilde{L}$:

$$\eta_{MN} A^N = A_M \tag{5.14}$$

as the way obvious with its index position, there is a natural isomorphism between $\tilde{L}$ and L^*. With this isomorphism at hand, we can identify these spaces. We note that the metric (5.13) implies that the inner product among $X, Y \in \Gamma(L)$ is $\langle X, Y \rangle = 0$ and the same is true even for $\tilde{L}$. This means that L and $\tilde{L}$ are maximally isotropic subbundles and $T\mathcal{M} = L \oplus \tilde{L}$.

Given these structures, one can define the space of multi-vectors $\hat{\mathcal{A}}^k(\mathcal{M})$ and the canonical projectors $\pi^{r,s}$. The projectors are defined, for example, as follows. The projectors in a para-complex manifold with $K = \mathrm{diag}(-1, +1)$, in their apparent representation, are given by

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.15}$$

We consider the case of $r = 2$, $s = 0$. The component expression of an element $T \in \hat{\mathcal{A}}^2(\mathcal{M})$ is denoted by

$$T^{MN} = \begin{pmatrix} t_{\mu\nu} & t_\mu{}^\nu \\ t^\mu{}_\nu & t^{\mu\nu} \end{pmatrix}. \tag{5.16}$$

The canonical projector $\pi^{2,0}$ defined through P is given by

$$P^M{}_K T^{KL} P^N{}_L = \begin{pmatrix} 0 & 0 \\ 0 & t^{\mu\nu} \end{pmatrix}. \tag{5.17}$$

Here, $t^{\mu\nu}$ is the element of $\mathcal{A}^{2,0}(\mathcal{M})$. This implies $\pi^{2,0}(T^{MN}) = t^{\mu\nu}$. The other projectors $\pi^{1,1}, \pi^{0,2}$ are defined similarly.

Now we define the Lie bracket on L. For $A, B \in \Gamma(L)$, this is given by

$$[A, B]_L = [A, B]_L^\mu \partial_\mu = (A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu)\partial_\mu. \tag{5.18}$$

Since this is the ordinary Lie bracket in differential geometry, it satisfies the Jacobi identity trivially. It is obvious that it also satisfies the Leibniz rule. With this bracket and the trivial bundle map $\rho_L = \mathrm{id}_L$ as the anchor, then L is endowed with a Lie algebroid structure. Note that since L is involutive with respect to $[\cdot,\cdot]_L$, it is integrable and define a Dirac structure on $T\mathcal{M}$. As discussed in Section 3.3, by introducing multi-vectors, we generalize the Lie bracket to the Schouten-Nijenhuis bracket. An explicit realization of the Schouten-Nijenhuis bracket in DFT is as follows. Given a k-vector $A \in \Gamma(\wedge^k L)$,

$$A = \frac{1}{k!} A^{\mu_1 \cdots \mu_k} \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_k}, \tag{5.19}$$

we introduce the "odd coordinate" $\zeta_\mu := \partial_\mu$. Then the k-vector is expressed as

$$A = \frac{1}{k!} A^{\mu_1 \cdots \mu_k} \zeta_{\mu_1} \cdots \zeta_{\mu_k}. \tag{5.20}$$

Note that ζ_μ can be treated as a Grassmann number whose differential $\partial/\partial\zeta_\mu$ is defined by the right derivative. Namely,

$$\frac{\partial}{\partial \zeta_{\mu_n}} (\zeta_{\mu_1} \cdots \zeta_{\mu_n} \cdots \zeta_{\mu_k}) = (\zeta_{\mu_1} \cdots \zeta_{\mu_n} \cdots \zeta_{\mu_k}) \frac{\overleftarrow{\partial}}{\partial \zeta_{\mu_n}} = (-1)^{k-n} \zeta_{\mu_1} \cdots \check{\zeta}_{\mu_n} \cdots \zeta_{\mu_k}. \tag{5.21}$$

Here the symbol $\check{\zeta}_{\mu_n}$ stands for that ζ_{μ_n} is removed. By using this ζ_μ derivative, the Schouten-Nijenhuis bracket is explicitly given by

$$[A, B]_{\mathrm{S}} = \left(\frac{\partial}{\partial \zeta_\mu} A \right) \partial_\mu B - (-1)^{(p-1)(q-1)} \left(\frac{\partial}{\partial \zeta_\mu} B \right) \partial_\mu A. \tag{5.22}$$

Here $A \in \Gamma(\wedge^p L), B \in \Gamma(\wedge^q L)$. The discussion is totally parallel in $\tilde{L}$. The same definition holds for $[\cdot,\cdot]_{\mathrm{S}}^*$ on $\tilde{L}$ where $\zeta_\mu = \partial_\mu$ is replaced by $\zeta^{*\mu} = \tilde{\partial}^\mu$. One can show that this expression satisfies the definition of the Schouten-Nijenhuis bracket discussed in Section 3.3. It is known that multi-vectors on a manifold define a Gerstenhaber algebra by the Schouten-Nijenhuis bracket [128]. By the Vaintrob theorem [129], a Lie algebroid structure over a vector bundle $V \to M$ and a Gerstenhaber algebra over multi-vectors $\Gamma(\wedge^\bullet V)$ are equivalent.

The symmetric pairing $\langle\!\langle \alpha, A \rangle\!\rangle$ is defined, for example,

$$\alpha(A_1, \cdots, A_s) = \alpha_{\mu_1 \cdots \mu_s} A_1^{\mu_1} \cdots A_s^{\mu_s}, \tag{5.23}$$

and so on. Since $\tilde{L}$ and L^* are identified via the natural isomorphism, the symmetric pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ is the inner product $\langle \cdot,\cdot \rangle$ in disguise. A Lie algebroid coboundary operator that maps a k-vector to a $(k+1)$-vector is given by the para-Dolbeault operator $\tilde{\mathrm{d}} : \wedge^k L \to \wedge^{k+1} L$. This is characterized by the following general relation:

$$\begin{aligned} \tilde{\mathrm{d}} X(\alpha_1, \ldots, \alpha_{k+1}) = & \sum_{i=1}^{k+1} (-1)^{i+1} \rho_{\tilde{L}}(\alpha_i) \cdot (X(\alpha_1, \ldots, \check{\alpha}_i, \ldots, \alpha_{k+1})) \\ & + \sum_{i<j} (-1)^{i+j} X([\alpha_i, \alpha_j]_{\mathrm{S}}^*, \alpha_1, \ldots, \check{\alpha}_i, \ldots, \check{\alpha}_j, \ldots, \alpha_{k+1}). \end{aligned} \tag{5.24}$$

Here $X \in \Gamma(\wedge^k L)$, $\alpha_i \in \Gamma(\tilde{L})$ and the symbol $\check{\alpha}_i$ stands for that the i-th α is removed. The bracket $[\cdot,\cdot]_{\mathrm{S}}^*$ and the anchor $\rho_{\tilde{L}}$ is defined on $\tilde{L}$. In particular, using the local coordinate, we find that the action of $\tilde{\mathrm{d}}$ on a k-vector X is explicitly given by

$$\tilde{\mathrm{d}}X = \frac{1}{k!}\tilde{\partial}^\mu X^{\nu_1\cdots\nu_k}(x,\tilde{x})\partial_\mu \wedge \partial_{\nu_1} \wedge \cdots \wedge \partial_{\nu_k}. \tag{5.25}$$

We confirm that this definition of $\tilde{\mathrm{d}}$ is compatible with the bracket $[\cdot,\cdot]_{\mathrm{S}}^*$. By the definition of $\tilde{\mathrm{d}}$ (5.24), for $k=1$, $A \in \Gamma(L)$, $\alpha_1,\alpha_2 \in \Gamma(\tilde{L})$ we have

$$\begin{aligned}\tilde{\mathrm{d}}A(\alpha_1,\alpha_2) &= (-1)^2\rho_{\tilde{L}}(\alpha_1)\cdot(A(\alpha_2)) + (-1)^3\rho_{\tilde{L}}(\alpha_2)\cdot(A(\alpha_1)) + (-1)^3 A([\alpha_1,\alpha_2]_{\mathrm{S}}^*)\\ &= \rho_{\tilde{L}}(\alpha_1)\cdot(A(\alpha_2)) - \rho_{\tilde{L}}(\alpha_2)\cdot(A(\alpha_1)) - A([\alpha_1,\alpha_2]_{\mathrm{S}}^*).\end{aligned} \tag{5.26}$$

Then in the DFT realization, since $\rho_{\tilde{L}}(\alpha_1) = \alpha_{1\mu}\tilde{\partial}^\mu$, we have

$$\begin{aligned}A([\alpha_1,\alpha_2]_{\mathrm{S}}^*) &= \rho_{\tilde{L}}(\alpha_1)\cdot(A(\alpha_2)) - \rho_{\tilde{L}}(\alpha_2)\cdot(A(\alpha_1)) - \tilde{\mathrm{d}}A(\alpha_1,\alpha_2)\\ &= \alpha_{1\mu}\tilde{\partial}^\mu(A^\nu\alpha_{2\nu}) - \alpha_{2\nu}\tilde{\partial}^\nu(A^\mu\alpha_{1\mu}) - (\tilde{\partial}^\mu A^\nu - \tilde{\partial}^\nu A^\mu)\alpha_{1\mu}\alpha_{2\nu}\\ &= A^\mu(\alpha_{1\nu}\tilde{\partial}^\nu\alpha_{2\mu} - \alpha_{2\nu}\tilde{\partial}^\nu\alpha_{1\mu}).\end{aligned} \tag{5.27}$$

Therefore we find that the exterior derivative $\tilde{\mathrm{d}}$ on L and the bracket $[\cdot,\cdot]_{\mathrm{S}}^*$ is compatible. The same discussion holds also for the operator d on the Lie algebroid $\tilde{L}$.

We next derive the Lie derivative in DFT. For $A, B \in \mathcal{A}^{1,0}(\mathcal{M})$ and $\alpha,\beta \in \mathcal{A}^{0,1}(\mathcal{M})$, the interior products (or the symmetric pairing) are realized as follows:

$$\begin{aligned}\iota_A\beta &= A^\mu\beta_\mu, &\quad \iota_A\mathrm{d}\beta &= (A^\nu\partial_\nu\beta_\mu - A^\nu\partial_\mu\beta_\nu)\tilde{\partial}^\mu,\\ \tilde{\iota}_\alpha B &= \alpha_\mu B^\mu, &\quad \tilde{\iota}_\alpha\tilde{\mathrm{d}}B &= (\alpha_\nu\tilde{\partial}^\nu B^\mu - \alpha_\nu\tilde{\partial}^\mu B^\nu)\partial_\mu.\end{aligned} \tag{5.28}$$

The Lie derivative defined in (5.6) is therefore given by

$$\begin{aligned}\mathcal{L}_A\beta &= (\mathrm{d}\iota_A + \iota_A\mathrm{d})\beta\\ &= \mathrm{d}(\iota_A\beta) + \iota_A(\mathrm{d}\beta) = \mathrm{d}(A^\nu\beta_\nu) + \iota_A(\partial_\mu\beta_\nu\tilde{\partial}^\mu \wedge \tilde{\partial}^\nu)\\ &= [(\partial_\mu A^\nu)\beta_\nu + A^\nu\partial_\mu\beta_\nu]\tilde{\partial}^\mu + A^\mu\partial_\mu\beta_\nu\tilde{\partial}^\nu - A^\nu\partial_\mu\beta_\nu\tilde{\partial}^\mu\\ &= (A^\nu\partial_\nu\beta_\mu + \beta_\nu\partial_\mu A^\nu)\tilde{\partial}^\mu.\end{aligned} \tag{5.29}$$

Similarly we have

$$\begin{aligned}\tilde{\mathcal{L}}_\alpha B &= (\tilde{\mathrm{d}}\tilde{\iota}_\alpha + \tilde{\iota}_\alpha\tilde{\mathrm{d}})B = \tilde{\mathrm{d}}(\alpha_\nu B^\nu) + \tilde{\iota}_\alpha(\tilde{\partial}^\mu B^\nu\partial_\mu \wedge \partial_\nu)\\ &= [(\tilde{\partial}^\mu\alpha_\nu)B^\nu + \alpha_\nu\tilde{\partial}^\mu B^\nu]\partial_\mu + \alpha_\mu\tilde{\partial}^\mu B^\nu\partial_\nu - \alpha_\nu\tilde{\partial}^\mu B^\nu\partial_\mu\\ &= (\alpha_\nu\tilde{\partial}^\nu B^\mu + B^\nu\tilde{\partial}^\mu\alpha_\nu)\partial_\mu.\end{aligned} \tag{5.30}$$

We have consistently defined the Lie algebroid $(\wedge^\bullet L, [\cdot,\cdot]_{\mathrm{S}}, \mathrm{d})$ and its dual Lie algebroid $(\wedge^\bullet\tilde{L},\ [\cdot,\cdot]_{\mathrm{S}}^*,\ \tilde{\mathrm{d}})$ in DFT.

We are now in a position to discuss doubled structures of $(L,\tilde{L})$. As we have discussed in Section 3.4, a Lie bialgebroid is defined by a Lie algebroid $(L,[\cdot,\cdot]_L,\rho_L,\mathrm{d})$ and its dual

Lie coalgebroid $(L^*, [\cdot,\cdot]_{L^*}, \rho_{L^*}, \mathrm{d}_*)$ together with a compatibility condition between them called the derivation condition (3.18). Again, this is given by

$$\mathrm{d}_*[X,Y]_{\mathrm{S}} = [\mathrm{d}_*X, Y]_{\mathrm{S}} + [X, \mathrm{d}_*Y]_{\mathrm{S}}, \qquad X, Y \in \Gamma(\wedge^\bullet L), \tag{5.31}$$

where $\mathrm{d} : \wedge^k L^* \to \wedge^{k+1} L^*$ and $\mathrm{d}_* : \wedge^k L \to \wedge^{k+1} L$ are exterior derivatives defined above. Now we examine the derivation condition in DFT by the explicit calculations. It is enough to show for $A, B \in \Gamma(L)$. The left hand side of (5.31) is given by

$$\begin{aligned}
\tilde{\mathrm{d}}[A,B]_{\mathrm{S}} &= \tilde{\partial}^\mu [A,B]^\nu_{\mathrm{S}} \partial_\mu \wedge \partial_\nu \\
&= \tilde{\partial}^\mu (A^\rho \partial_\rho B^\nu - B^\rho \partial_\rho A^\nu) \partial_\mu \wedge \partial_\nu \\
&= (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu + A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu - \tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu,
\end{aligned} \tag{5.32}$$

while the right hand side is calculated by using the explicit form of the Schouten-Nijenhuis bracket:

$$\begin{aligned}
[\tilde{\mathrm{d}}A, B]_{\mathrm{S}} &= \left(\frac{\partial}{\partial \zeta_\rho} \tilde{\mathrm{d}} A\right) \partial_\rho B - (-1)^0 \left(\frac{\partial}{\partial \zeta_\rho} B\right) \partial_\rho \tilde{\mathrm{d}} A \\
&= (\tilde{\partial}^\mu A^\rho \zeta_\mu - \tilde{\partial}^\rho A^\mu \zeta_\mu) \partial_\rho B^\nu \zeta_\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu \zeta_\mu \zeta_\nu \\
&= (\tilde{\partial}^\mu A^\rho \partial_\rho B^\nu - \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu - B^\rho \partial_\rho \tilde{\partial}^\mu A^\nu) \partial_\mu \wedge \partial_\nu, \\
[A, \tilde{\mathrm{d}}B]_{\mathrm{S}} &= -[\tilde{\mathrm{d}}B, A]_{\mathrm{S}} \\
&= -(\tilde{\partial}^\mu B^\rho \partial_\rho A^\nu - \tilde{\partial}^\rho B^\mu \partial_\rho A^\nu - A^\rho \partial_\rho \tilde{\partial}^\mu B^\nu) \partial_\mu \wedge \partial_\nu.
\end{aligned} \tag{5.33}$$

From these expressions, we obtain

$$\tilde{\mathrm{d}}[A,B]_{\mathrm{S}} = [\tilde{\mathrm{d}}A, B]_{\mathrm{S}} + [A, \tilde{\mathrm{d}}B]_{\mathrm{S}} + (\tilde{\partial}^\rho A^\mu \partial_\rho B^\nu + \tilde{\partial}^\rho B^\nu \partial_\rho A^\mu) \partial_\mu \wedge \partial_\nu. \tag{5.34}$$

The last contribution represents the violation of the derivation condition (5.31). We have then explicitly shown that given the Lie algebroid structures L and $\tilde{L} \simeq L^*$ in DFT, they do not form a Lie bialgebroid in general. Although this is true, following the general discussion in Section 3.5, the double $L \oplus L^*$ defines a Vaisman algebroid. The anchor in the Vaisman algebroid is defined as $\rho_{\mathrm{V}} = \rho_L + \rho_{L^*}$ while the bilinear form (Ξ_1, Ξ_2) for $\Xi_i \in \Gamma(T\mathcal{M})$ is given by

$$(\Xi_1, \Xi_2) = (A + \alpha, B + \beta) = \frac{1}{2}\Big\{\langle\!\langle \alpha, B \rangle\!\rangle + \langle\!\langle \beta, A \rangle\!\rangle\Big\}. \tag{5.35}$$

Here $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the symmetric pairing defined before. The differential operator is defined as $\mathcal{D} = \mathrm{d} + \tilde{\mathrm{d}}$. By using the Lie brackets $[\cdot,\cdot]_L$, $[\cdot,\cdot]_{L^*}$, Lie derivatives $\mathcal{L}_A$, $\tilde{\mathcal{L}}_\alpha$ and operators $\mathrm{d}, \iota, \tilde{\mathrm{d}}, \tilde{\iota}$, we define the Vaisman bracket for vectors $\Xi_i \in \Gamma(T\mathcal{M})$:

$$\begin{aligned}
[\Xi_1, \Xi_2]_{\mathrm{V}} = [A + \alpha, B + \beta]_{\mathrm{V}} &= [A, B]_L + \mathcal{L}_A \beta - \mathcal{L}_B \alpha - \frac{1}{2} \mathrm{d}(\iota_A \beta - \iota_B \alpha) \\
&\quad + [\alpha, \beta]_{\tilde{L}} + \tilde{\mathcal{L}}_\alpha B - \tilde{\mathcal{L}}_\beta A - \frac{1}{2} \tilde{\mathrm{d}}(\tilde{\iota}_\alpha B - \tilde{\iota}_\beta A),
\end{aligned} \tag{5.36}$$

This is nothing but the C-bracket (2.23). The quadruple $(L\oplus\tilde{L},[\cdot,\cdot]_{\mathsf{C}},\rho_{\mathrm{V}},(\cdot,\cdot))$ then defines a Vaisman algebroid.

We note that the last term in (5.34) is rewritten as

$$\tilde{\partial}^{\rho}A^{\mu}\partial_{\rho}B^{\nu}+\tilde{\partial}^{\rho}B^{\nu}\partial_{\rho}A^{\mu}=\eta^{KL}\partial_{K}A^{\mu}\partial_{L}B^{\nu}. \tag{5.37}$$

It is obvious that this vanishes when the strong constraint is imposed. This means that the derivation condition between L and $\tilde{L}$ is satisfied and $(L,\tilde{L})$ becomes a Lie bialgebroid when the strong constraint is imposed and the gauge transformation parameters are restricted [93]. In this case, the double $L\oplus\tilde{L}$ defines a Courant algebroid following the general discussions [102, 130]. This completely agrees with the analysis in [116] where the pre-DFT algebroid (Vaisman algebroid) becomes a Courant algebroid after imposing the strong constraint. We again stress that an algebraic origin of the strong constraint is the derivation condition that is a compatibility condition between L and $\tilde{L}$ which allows them to be a Lie bialgebroid.

5.3 Relations with generalized geometry

In this subsection, we discuss the gauge symmetries associated with $L,\tilde{L}$ and the relation to generalized geometry. As discussed in [73, 131], the structure of the C-bracket in DFT naturally arises as a Vaisman bracket on a para-Hermitian geometry. The C-bracket is recognized as a T-duality covariantized Lie bracket-like structure that accommodates the diffeomorphism and the B-field gauge symmetry algebra in the NSNS sector of supergravity. The geometric realization of the C-bracket does not necessarily require the strong constraint. In this sense, the C-bracket governs the "off-shell" gauge symmetry of DFT (a symmetry without the strong constraint). Due to the para-complex structure underlying the doubled space-time $\mathcal{M}$, there is a natural decomposition of the tangent bundle $T\mathcal{M}=L\oplus\tilde{L}$ in which Lie algebroid structures are found. Since the distributions $L,\tilde{L}$ are Dirac structures and therefore are integrable, they are given by tangent bundles of foliations $\mathcal{F},\tilde{\mathcal{F}}$ in $\mathcal{M}$. A physical space-time is therefore identified as a leaf defined by $\tilde{x}_{\mu}=$ const. in a para-Hermitian manifold. With the natural isomorphism induced by an inner product defined by the metric η, the vector components in $\tilde{L}=T\tilde{\mathcal{F}}\simeq L^{*}=T^{*}\mathcal{F}$ is identified with 1-forms over a leaf in $\mathcal{M}$. Therefore, one can understand that the Lie bracket $[\cdot,\cdot]_{L}$ over L governs the diffeomorphism parametrized by vector gauge parameters ξ_i^{μ} while the bracket $[\cdot,\cdot]_{\tilde{L}}$ over $\tilde{L}$ represents the B-field gauge symmetry parametrized by 1-forms $\tilde{\xi}_{i,\mu}$. Since the Lie bracket for the 1-forms $[\tilde{\xi}_1,\tilde{\xi}_2]_{\tilde{L}}$ is generically non-zero, the T-duality covariantized B-field gauge symmetry is effectively enhanced to non-Abelian "off-shell".

Upon the imposition of the strong constraint, the gauge algebra is closed by the C-bracket. Therefore in order that the algebra given by the C-bracket generates a symmetry, the strong constraint is necessarily satisfied, ether implicit or explicitly. A way to solve the strong constraint trivially is to make the winding derivative be vanishing $\tilde{\partial}*=0$. In this case, the bracket including 1-forms vanish $[\tilde{\xi}_1,\tilde{\xi}_2]_{\tilde{L}}=0$ and the C-bracket is reduced

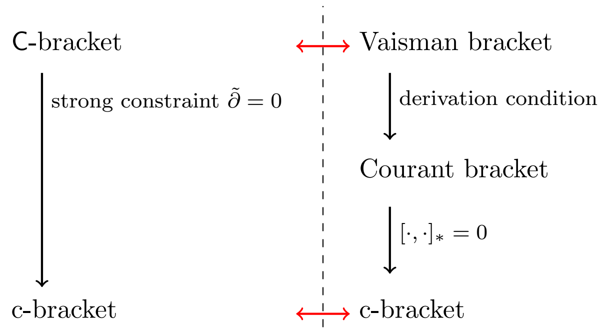

Figure 5.1: Paths to the c-bracket in DFT and Vaisman algebroids.

to the c-bracket defined in (3.26). This means that by imposing the section condition on any DFT fields and gauge parameters and making the theory be "on-shell" (*i.e.* defined on a physical subspace), the non-Abelian "off-shell" B-field gauge symmetry becomes an Abelian symmetry "on-shell". In this sense, the c-bracket is an "on-shell" counterpart of the C-bracket. From a mathematical point of view, the c-bracket is obtained by first imposing the derivation condition (3.18) on the Vaisman bracket and then make the Lie bracket on $\tilde{L}$ be a zero-bracket $[\cdot,\cdot]_{\tilde{L}} = 0$ (see Fig 5.1). As we have explicitly shown, with the adaptation of the derivation condition, $(L, \tilde{L})$ forms a Lie bialgebroid. Through the prescription by Liu-Weinstein-Xu [102], the c-bracket defines a Courant algebroid. This c-bracket is nothing but the original Courant bracket appeared in generalized geometry [37].

Given a para-Dolbeault cohomology, the "on-shell" fields and gauge parameters in DFT satisfying the strong constraint are characterized by para-holomorphic quantities defined by the para-Dolbeault operators:

$$\text{para-holomorphic}: \qquad \tilde{\mathrm{d}}\Phi = 0, \tag{5.38}$$

where Φ is any doubled fields and gauge parameters. This is equivalent to say that the para-holomorphic quantities are restricted in leaves in the foliation $\mathcal{F}$. We note that this is not the unique solution to the strong constraint. The other possibility

$$\text{anti-para-holomorphic}: \qquad \mathrm{d}\Phi = 0, \tag{5.39}$$

also satisfies the strong constraint trivially. The anti-para-holomorphic quantities are defined along the transverse directions to leaves. Namely, they live in the winding space defined by $x^\mu = \text{const}$. We note that this kind of winding dependent space-time actually appears in solutions to DFT [132–136].

5.4 Ante-Courant algebroid on para-Hermitian manifold

Recalling the discussion in Chapter 3, it is clear that the algebroids by C-bracket are not only Vaisman and Courant algebroid, but also pre- Courant and Ante-Courant algebroids. In this section, we realise these algebroids on para-Hermitian geometry. What conditions are required to construct these algebroids?

By the general discussion in Chapter 3, the doubled structure of an ante-Courant algebroid is compatible with the C-bracket when the anchor satisfies the equation (3.72). We examine this condition in the para-Hermitian manifold $\mathcal{M}$. Since E and E^* in the general discussion correspond to L and $\tilde{L}$ in the para-Hermitian manifold, we first write down the anchor structures in each Lie algebroid. The anchor $\rho_L : L \to T\mathcal{M}$ on the Lie algebroid L is expressed as

$$\begin{aligned}\rho_L(X) &= (\rho_L)^M{}_\nu X^\nu \partial_M \\ &= \rho^\mu{}_\nu X^\nu \partial_\mu + \rho_{\mu\nu} X^\nu \tilde{\partial}^\mu,\end{aligned} \tag{5.40}$$

where $X \in \Gamma(L)$. Note that the target of ρ_L is $T\mathcal{M}$. The adjoint ρ_L^* is defined through the following relation

$$\begin{aligned}&\langle q, \rho_L(X)\rangle \\ &= (\rho^t)_\mu{}^\nu \eta_\nu X^\mu + (\rho^t)_{\nu\mu} Y^\mu X^\nu \\ &= \langle \rho_L^*(q), X\rangle,\end{aligned} \tag{5.41}$$

where $q = \eta + Y \in \Gamma(T^*\mathcal{M})$ and the symbol t means transposition of a matrix. From this expression, we write $(\rho_L^*)_\mu{}^N = ((\rho^t)_\mu{}^\nu, (\rho^t)_{\mu\nu})$. Likewise, the anchor $\rho_{\tilde{L}} : \tilde{L} \to T\mathcal{M}$ on $\tilde{L}$ and its adjoint are expressed as $(\rho_{\tilde{L}})^{M\nu} = (\tilde{\rho}^{\mu\nu}, \tilde{\rho}_\mu{}^\nu)$, $(\rho_{\tilde{L}}^*)^{\mu N} = ((\tilde{\rho}^t)^{\mu\nu}, (\tilde{\rho}^t)^\mu{}_\nu)$. Therefore, the anchor $\rho_{\mathrm{V}} = \rho_L + \rho_{\tilde{L}}$ on $T\mathcal{M} = L \oplus \tilde{L}$ is given by

$$(\rho_{\mathrm{V}})^M{}_N = \begin{pmatrix} \rho^\mu{}_\nu & \tilde{\rho}^{\mu\nu} \\ \rho_{\mu\nu} & \tilde{\rho}_\mu{}^\nu \end{pmatrix}. \tag{5.42}$$

The component expression of $\rho_L\rho_{\tilde{L}}^* + \rho_{\tilde{L}}\rho_L^*$ is

$$\left(\rho_L\rho_{\tilde{L}}^* + \rho_{\tilde{L}}\rho_L^*\right)^{MN} = \begin{pmatrix} \rho^\mu{}_\sigma(\tilde{\rho}^t)^{\sigma\nu} + (\tilde{\rho})^{\mu\sigma}(\rho^t)_\sigma{}^\nu & \rho^\mu{}_\sigma(\tilde{\rho}^t)^\sigma{}_\nu + (\tilde{\rho})^{\mu\sigma}(\rho^t)_{\sigma\nu} \\ \rho_{\mu\sigma}(\tilde{\rho}^t)^{\sigma\nu} + (\tilde{\rho})_\mu{}^\sigma(\rho^t)_\sigma{}^\nu & \rho_{\mu\sigma}(\tilde{\rho}^t)^\sigma{}_\nu + (\tilde{\rho})_\mu{}^\sigma(\rho^t)_{\sigma\nu} \end{pmatrix} \tag{5.43}$$

Since d_0 is the exterior derivative on $T\mathcal{M}$, this is given by $\mathrm{d}_0 f = \partial_M f \mathrm{d}x^M = \partial_\mu f \mathrm{d}x^\mu + \tilde{\partial}^\mu f \mathrm{d}\tilde{x}_\mu \in \Gamma(T^*\mathcal{M})$ for $f \in C^\infty(\mathcal{M})$ and the condition (5.43) is expressed by

$$\left(\rho_L\rho_{\tilde{L}}^* + \rho_{\tilde{L}}\rho_L^*\right)(\mathrm{d}_0 f)\cdot g = \left(\rho_L\rho_{\tilde{L}}^* + \rho_{\tilde{L}}\rho_L^*\right)^{MN}\partial_M f \partial_N g = 0. \tag{5.44}$$

Now, we consider a concrete example of ρ_{V} in (5.42) in the flat para-Hermitian manifold. The most natural candidate of the anchor is given by the diagonal form

$$(\rho_{\mathrm{V}})^M{}_N = \begin{pmatrix} \rho^\mu{}_\nu & 0 \\ 0 & \tilde{\rho}_\mu{}^\nu \end{pmatrix}. \tag{5.45}$$

In particular, the simplest example is $(\rho_L)^M{}_\nu = (\delta^\mu{}_\nu, 0)$, $(\rho_{\tilde{L}})_\mu{}^N = (0, \delta_\mu{}^\nu)$. In this case, the condition (5.44) is given by

$$\begin{aligned}0 = (\rho_*\rho^* + \rho\rho_*^*)^{MN}\partial_N f \partial_M g &= \left(\partial_\nu f, \tilde{\partial}^\nu f\right)\begin{pmatrix} 0 & \rho^\mu{}_\sigma(\tilde{\rho})^\sigma{}_\nu \\ \tilde{\rho}_\mu{}^\sigma(\rho^t)_\sigma{}^\nu & 0 \end{pmatrix}\begin{pmatrix} \partial_\nu g \\ \tilde{\partial}^\nu g \end{pmatrix} \\ &= \eta^{MN}\partial_M f \partial_N g.\end{aligned} \tag{5.46}$$

Here η^{MN} is the $O(D,D)$ invariant metric in DFT. That is, the condition for an ante-Courant algebroid on a flat para-Hermitian manifold $\mathcal{M}$ is nothing but the strong constraint only for functions f,g. Since the strong constraint is given by $\eta^{MN}\partial_M\Psi\partial_N\Phi = 0$ for any quantities Ψ,Φ in the para-Hermitian manifold $\mathcal{M}$, the condition (5.46) is a relaxed version of the constraint. This is consistent with the result in [116].

5.5 Pre-Courant algebroid on para-Hermitian manifold

Next, we examine the conditions for a pre-Courant algebroid on $\mathcal{M}$. As we have clarified in Section 3, the condition is only (3.68). We write down this condition in the flat para-Hermitian manifold. In particular, when ρ_{V} is given by (5.45) and $\rho^\mu{}_\nu = \delta^\mu{}_\nu$, $\tilde{\rho}_\mu{}^\nu = \delta_\mu{}^\nu$, we have $X = X^\mu\partial_\mu \in \Gamma(L), \mathrm{d}_*f = \tilde{\partial}^\mu f\partial_\mu, \mathrm{d}f = \partial_\mu f\tilde{\partial}^\mu$ [85]. Therefore, the first condition in (3.68) is found to be

$$\begin{aligned} 0 &= \mathcal{L}_{\mathrm{d}f}X + [\mathrm{d}_*f, X] \\ &= \partial_\nu f\tilde{\partial}^\nu X^\mu\partial_\mu + \tilde{\partial}^\nu f\partial_\nu X^\mu\partial_\mu \\ &= \eta^{MN}\partial_M f\partial_N X^\mu\partial_\mu. \end{aligned} \tag{5.47}$$

The same is true for $\xi \in \Gamma(\tilde{L})$. The second condition in (3.68) is

$$\eta^{MN}\partial_M f\partial_N\xi_\mu\tilde{\partial}^\mu = 0. \tag{5.48}$$

Since any pre-Courant algebroids are ante-Courant algebroids, the condition (5.46) is also satisfied. These conditions (5.47) and (5.48) are nothing but the strong constraint for $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(L)$ and $\xi \in \Gamma(\tilde{L})$. This is again a relaxed version of the strong constraint in DFT.

As noted in Chapter 3, (5.48) is the necessary condition for a Poisson structure. In this case, we have the following bracket,

$$\{g,f\} = \tilde{\partial}^\mu g\partial_\mu f = -\tilde{\partial}^\mu f\partial_\mu g = -\{f,g\}. \tag{5.49}$$

We find that the skew-symmetric nature is guaranteed by the condition (5.46). The Jacobiator of the bracket is calculated as

$$\begin{aligned} \mathrm{Jac}(f,g,h) = {}& \tilde{\partial}^\mu(\tilde{\partial}^\nu f)\partial_\nu g\partial_\mu h + \tilde{\partial}^\nu f\tilde{\partial}^\mu(\partial_\nu g)\partial_\mu h + \tilde{\partial}^\mu(\tilde{\partial}^\nu g)\partial_\nu h\partial_\mu f \\ & + \tilde{\partial}^\nu g\tilde{\partial}^\mu(\partial_\nu h)\partial_\mu f + \tilde{\partial}^\mu(\tilde{\partial}^\nu h)\partial_\nu f\partial_\mu g + \tilde{\partial}^\nu h\tilde{\partial}^\mu(\partial_\nu f)\partial_\mu g. \end{aligned} \tag{5.50}$$

Due to the conditions (5.47) and (5.48), we find $\mathrm{Jac}(f,g,h) = 0$ and confirm that (5.49) indeed defines a Poisson structure.

5.6 Twisted algebroids

We have been focusing on the doubled structures of the algebroids and we clarified the conditions for vectors, forms and functions in the doubled space-time. One can introduce additional structures known as the twist by background fluxes in the manifold. For example, the twist of the standard Courant algebroid defined by the generalized tangent bundle $TM \oplus T^*M$ over M has been discussed [109]. The background 3-form H modifies the standard Courant bracket $[\cdot,\cdot]_{\mathrm{c}}$ giving a new bracket $[\cdot,\cdot]_H$:

$$\begin{aligned}
&[X_1+\xi_1, X_2+\xi_2]_{\mathrm{c}} = [X_1, X_2] + \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \frac{1}{2}\mathrm{d}(\langle \xi_1 X_2\rangle - \langle \xi_2 X_1\rangle),\\
&[X_1+\xi_1, X_2+\xi_2]_H = [X_1+\xi_1, X_2+\xi_2]_{\mathrm{c}} + \iota_{X_2}\iota_{X_1} H,
\end{aligned} \tag{5.51}$$

where, $X_i \in \Gamma(TM), \xi_i \in \Gamma(T^*M)$. The twisted bracket $[\cdot,\cdot]_H$ preserves the Courant algebroid structure when $\mathrm{d}H = 0$ [42]. Physically, this 3-form H corresponds to the H-flux that appears in the NS-NS sector of type II supergravity. It is known that this H-flux is related to the other fluxes f, Q, R in type II string theory via the T-duality transformations. This is represented by the following form [137]:

$$H_{abc} \overset{T_c}{\longleftrightarrow} f_{ab}{}^{c} \overset{T_b}{\longleftrightarrow} Q_a{}^{bc} \overset{T_a}{\longleftrightarrow} R^{abc}. \tag{5.52}$$

The twist of the Courant algebroid with the C-bracket has been discussed in [116, 138]. In this section, we study compatibility conditions for the doubled and the twisted structures of the other DFT algebroids.

In order to introduce the twist structure, we consider a doubled $(2,1)$-tensor $F = F_{MN}{}^{L}\mathrm{d}x^M \otimes \mathrm{d}x^N \otimes \partial_L$ on a flat para-Hermitian manifold $\mathcal{M}$. We prefer to use a $(2,1)$-tensor rather than a 3-form on $\mathcal{M}$ because we never introduce "the generalized doubled tangent bundle" over $\mathcal{M}$. We then define the twisted C-bracket $[\cdot,\cdot]_{\mathrm{F}}$ as follows:

$$[e_1, e_2]_{\mathrm{F}} = [e_1, e_2]_{\mathsf{C}} + \iota_{e_2}\iota_{e_1} F, \quad e_i \in \Gamma(\mathcal{V}). \tag{5.53}$$

Here $\iota_{e_i} : \Gamma(\wedge^p \mathcal{V}^*) \to \Gamma(\wedge^{p-1}\mathcal{V}^*)$ is the interior product defined by

$$(\iota_{e_i} q)(a_1, \ldots, a_{p-1}) = h(e_i, a_1, \ldots, a_{p-1}), \quad q \in \Gamma(\wedge^p \mathcal{V}^*),\ a_1, \cdots, a_{p-1} \in \Gamma(\mathcal{V}). \tag{5.54}$$

In the following, we assume that $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ is a Vaisman algebroid with the doubled structure discussed in the previous sections.

Twisted Vaisman algebroid

We consider $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{F}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$. and examine Axiom V1 (C3) and Axiom V2 (C5). Axiom V1, the equation (V1), gives the Leibniz rule for the bracket. Expanding the left-hand side of the equation (V1), we find

$$\begin{aligned}
[e_1, fe_2]_{\mathrm{F}} &= [e_1, fe_2]_{\mathsf{C}} + \iota_{e_2}\iota_{e_1} F\\
&= f[e_1, e_2]_{\mathsf{C}} + f\iota_{e_2}\iota_{e_1} F + (\rho_{\mathrm{V}}(e_1)f)e_2 - (e_1, e_2)_+ \mathcal{D}f\\
&= f[e_1, e_2]_{\mathrm{F}} + (\rho_{\mathrm{V}}(e_1)f)e_2 - (e_1, e_2)_+ \mathcal{D}f,
\end{aligned} \tag{5.55}$$

where we have used the property of the C-bracket. Therefore, $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{F}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ satisfies Axiom V1 automatically.

Axiom V2, the equation (V2), is a compatibility condition between $(\cdot,\cdot)_+$ and $\rho_{\rm V}$. The right-hand side of the equation (V2) is evaluated as

$$\begin{aligned}&([e_1,e_2]_{\rm F}+\mathcal{D}(e_1,e_2)_+,e_3)_+ + (e_2,[e_1,e_3]_{\rm F}+\mathcal{D}(e_1,e_3)_+)_+\\&=([e_1,e_2]_{\rm V}+\mathcal{D}(e_1,e_2)_+,e_3)_+ + (e_2,[e_1,e_3]_{\rm V}+\mathcal{D}(e_1,e_3)_+)_+\\&\quad+(\iota_{e_2}\iota_{e_1}F,e_3)_+ + (e_2,\iota_{e_3}\iota_{e_1}F)_+.\end{aligned}\tag{5.56}$$

The first and the second terms on the right-hand side become $\rho_{\rm V}(e_1)(e_2,e_3)_+$. Thus, Axiom V2 is satisfied when the third and the fourth terms vanish:

$$(\iota_{e_2}\iota_{e_1}F,e_3)_+ + (e_2,\iota_{e_3}\iota_{e_1}F)_+ = 0 \quad {}^\forall e_1,e_2,e_3\in\Gamma(\mathcal{V}).\tag{5.57}$$

Since the basis of $\Gamma(\mathcal{V})$ is $\partial_M=(\partial_\mu,\tilde{\partial}^\mu)$, $\iota_{e_2}\iota_{e_1}F$ is given by

$$\iota_{e_2}\iota_{e_1}F=(e_1)^M(e_2)^N F_{MN}{}^l\partial_l+(e_1)^M(e_2)^N F_{MNl}\tilde{\partial}^l.\tag{5.58}$$

Then, we can denote $(\cdot,\cdot)_+$ as

$$(e_1,e_2)_+=\frac{1}{2}(\langle\xi_1,X_2\rangle+\langle\xi_2,X_1\rangle)=\frac{1}{2}\eta_{MN}(e_1)^M(e_2)^N.\tag{5.59}$$

Here, $X_i\in\Gamma(L),\xi_i\in\Gamma(\tilde{L}^*)$ and $e_i=X_i+\xi_i$. Therefore, the condition (5.57) becomes

$$\begin{aligned}0&=(\iota_{e_2}\iota_{e_1}F,e_3)_+ + (e_2,\iota_{e_3}\iota_{e_1}F)_+\\&=\frac{1}{2}(\eta_{KL}F_{MN}{}^K+\eta_{NK}F_{ML}{}^K)(e_1)^M(e_2)^N(e_3)^L.\end{aligned}\tag{5.60}$$

Namely,

$$F_{MNL}+F_{MLN}=0.\tag{5.61}$$

Here the doubled indices are raised and lowered by the $O(D,D)$ invariant metric η_{MN} and its inverse η^{MN}. In summary, $(\mathcal{V},[\cdot,\cdot]_{\rm F},\rho_{\rm V},(\cdot,\cdot)_+)$ becomes a twisted Vaisman algebroid only when the condition (5.61) is satisfied. This means that the tensor F_{MNL} is anti-symmetric with respect to the latter two indices. We note that the doubled tensor $F_{MN}{}^K$ is decomposed as $F_{MN}{}^K=(H_{\mu\nu\rho},f_{\mu\nu}{}^\rho,Q_\mu{}^{\nu\rho},R^{\mu\nu\rho})$ involving all the fluxes in (5.52).

Twisted ante-Courant algebroid

Next we discuss a twisted ante-Courant algebroid with the doubled structure. We assume that $(\mathcal{V},[\cdot,\cdot]_{\rm C},\rho_{\rm V},(\cdot,\cdot)_+)$ is an ante-Courant algebroid and look for conditions that $(\mathcal{V},[\cdot,\cdot]_{\rm F},\rho_{\rm V},(\cdot,\cdot)_+)$ becomes also an ante-Courant algebroid. Since any ante-Courant algebroids are Vaisman algebroids, the tensor F should satisfy the condition (5.61). As we have discussed, the condition for the ante-Courant algebroid is (3.72). However, this is the condition for the anchor map which is irrelevant to the bracket structure. Therefore, we need no extra conditions for F.

Twisted pre-Courant algebroid

Next, we discuss a twisted pre-Courant algebroid. Again we assume that $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ is a pre-Courant algebroid. We write down the conditions that $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{F}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ becomes a pre-Courant algebroid. In addition to the condition (5.61), we need Axiom C2, namely, the homomorphism of ρ_{V} (3.20). The left-hand side of the equation in (3.20) is evaluated as

$$\begin{aligned}\rho_{\mathrm{V}}([e_1,e_2]_{\mathrm{F}}) &= \rho_{\mathrm{V}}([e_1,e_2]_{\mathrm{V}} + \iota_{e_2}\iota_{e_1}F) \\ &= [\rho_{\mathrm{V}}(e_1), \rho_{\mathrm{V}}(e_2)] + \rho_{\mathrm{V}}\big(\iota_{e_2}\iota_{e_1}F\big).\end{aligned} \tag{5.62}$$

Thus, the condition is

$$\rho_{\mathrm{V}}\big(\iota_{e_2}\iota_{e_1}F\big) = (e_1)^M(e_2)^N(\rho_{\mathrm{V}})^L{}_K F_{MN}{}^K\partial_L = 0. \tag{5.63}$$

In component, we have,

$$(\rho_{\mathrm{V}})^L{}_K F_{MN}{}^K = 0. \tag{5.64}$$

Then for the non-zero tensor F, the anchor should satisfy

$$\det \rho_{\mathrm{V}} = 0. \tag{5.65}$$

Therefore, we should add the condition not only for the tensor (5.61) but also for the anchor (5.65) to obtain a twisted pre-Courant algebroid $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{F}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$. In particular, when (5.45) is adapted for ρ_{V}, either ρ_L or $\rho_{\tilde{L}}$ must be zero.

Twisted Courant algebroid

Finally, we consider a twisted Courant algebroid. We assume that $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ is a Courant algebroid. We calculate the Jacobiator (3.19) for $[e_1,e_2]_{\mathrm{F}}$ and confirm Axiom C1. The result is

$$\begin{aligned}[[e_1,e_2]_{\mathrm{F}}, e_3]_{\mathrm{F}} + \text{c.p.} &= \mathcal{D}T_{\mathrm{F}}(e_1,e_2,e_3) \\ &\quad - \frac{1}{3}\mathcal{D}\big((\iota_{e_2}\iota_{e_1}F, e_3)_+ + \text{c.p.}\big) \\ &\quad + \big(\iota_{e_3}\iota_{([e_1,e_2]_{\mathrm{V}})}F + [\iota_{e_2}\iota_{e_1}F, e_3]_{\mathrm{V}} + \iota_{e_3}\iota_{(\iota_{e_2}\iota_{e_1}F)}F + \text{c.p.}\big).\end{aligned} \tag{5.66}$$

Here T_{F} is defined by

$$T_{\mathrm{F}}(e_1,e_2,e_3) = T(e_1,e_2,e_3) + \frac{1}{3}\big((\iota_{e_2}\iota_{e_1}F, e_3)_+ + \text{c.p.}\big). \tag{5.67}$$

Then the condition for Axiom C1 is

$$\begin{aligned}&-\frac{1}{3}\mathcal{D}\big((\iota_{e_2}\iota_{e_1}F, e_3)_+ + \text{c.p.}\big) \\ &+ \big(\iota_{e_3}\iota_{([e_1,e_2]_{\mathrm{V}})}F + [\iota_{e_2}\iota_{e_1}F, e_3]_{\mathrm{V}} + \iota_{e_3}\iota_{(\iota_{e_2}\iota_{e_1}F)}F + \text{c.p.}\big) = 0.\end{aligned} \tag{5.68}$$

Since $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{F}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ should be a pre-Courant algebroid, the anchor satisfies the condition $\det \rho_{\mathrm{V}} = 0$. To solve this condition, we assume that the anchor is given by the diagonal form (5.45). Then one of the anchors ρ or $\tilde{\rho}$ must be zero. In the following, we select a frame where $\tilde{\rho} = 0$ and ρ is an identity matrix. When $\rho_{\tilde{L}} = 0$, this implies $\mathrm{d}_* f = 0$ and $\mathcal{L}_{\xi_i} = 0$ [108]. Since we assumed that $(\mathcal{V}, [\cdot,\cdot]_{\mathsf{C}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ becomes a Courant algebroid, the derivation condition (3.18) must hold. This is represented by the local coordinate as [85]

$$0 = \tilde{\partial}^\rho A^\mu \partial_\rho B^\nu + \tilde{\partial}^\rho B^\nu \partial_\rho A^\mu = \eta^{KL} \partial_K A^\mu \partial_L B^\nu, \quad A, B \in \Gamma(L). \tag{5.69}$$

Furthermore, the "relaxed" version of the strong constraints (5.46), (5.47) and (5.48) are satisfied as discussed before. The most natural solution to these conditions is obtained by setting $\tilde{\partial}^\mu \Psi = 0$ for any quantities Ψ in $\mathcal{M}$. Then the Lie bracket $[\cdot,\cdot]_*$ becomes zero.

With these conditions at hand, we evaluate (5.68). The first term in the left-hand side of (5.68) is expressed as

$$\mathcal{D}(\iota_{e_2}\iota_{e_1} F, e_3)_+ = \frac{1}{2}\mathcal{D}(\iota_{q_3}\iota_{e_2}\iota_{e_1} F). \tag{5.70}$$

Here we have introduced the interior product $\iota_{q_i} : \Gamma(\wedge^p \mathcal{V}) \to \Gamma(\wedge^{p-1}\mathcal{V})$ by a doubled 1-form q_i which acts on a doubled k-vector. Therefore,

$$-\frac{1}{3}\mathcal{D}((\iota_{e_2}\iota_{e_1} F, e_3)_+ + \text{c.p.}) = -\frac{1}{3}\mathrm{d}(\iota_{q_3}\iota_{e_2}\iota_{e_1} F + \text{c.p.}), \tag{5.71}$$

where we have used the fact $\mathcal{D} = \mathrm{d} + \mathrm{d}_*$. Likewise, the second term in the left-hand side in (5.68) becomes

$$\iota_{e_3}\iota_{[e_1,e_2]_{\mathsf{C}}} F + \text{c.p.} = \iota_{e_3}\iota_{([X_1,X_2]_L + \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \mathrm{d}(e_1,e_2)_-)} F + \text{c.p.} \tag{5.72}$$

The third term in the left-hand side of (5.68) is

$$\begin{aligned} [\iota_{e_2}\iota_{e_1} F, e_3]_{\mathrm{V}} + \text{c.p.} = {} & [\iota_{e_2}\iota_{e_1} F^\mu \partial_\mu, X_3] + \mathcal{L}_{(\iota_{e_2}\iota_{e_1} F^\mu \partial_\mu)} \xi_3 \\ & - \mathcal{L}_{X_3}(\iota_{e_2}\iota_{e_1} F_\mu \tilde{\partial}^\mu) + \mathrm{d}(\iota_{e_2}\iota_{e_1} F, e_3)_- + \text{c.p.} \end{aligned} \tag{5.73}$$

Here we have used the following notations,

$$\begin{aligned} \iota_{e_2}\iota_{e_1} F^l \partial_l &= (e_1)^M (e_2)^N F_{MN}{}^l \partial_l, \\ \iota_{e_2}\iota_{e_1} F_l \tilde{\partial}^l &= (e_1)^M (e_2)^N F_{MNl} \tilde{\partial}^l. \end{aligned} \tag{5.74}$$

Then the condition (5.68) is found to be

$$\begin{aligned} & -\frac{1}{3}\mathrm{d}(\iota_{q_3}\iota_{e_2}\iota_{e_1} F) + \iota_{e_3}\iota_{([X_1,X_2]_L + \mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1 + \mathrm{d}(e_1,e_2)_-)} F + [\iota_{e_2}\iota_{e_1} F^\mu \partial_\mu, X_3] \\ & + \mathcal{L}_{(\iota_{e_2}\iota_{e_1} F^\mu \partial_\mu)} \xi_3 - \mathcal{L}_{X_3}(\iota_{e_2}\iota_{e_1} F_\mu \tilde{\partial}^\mu) + \mathrm{d}(\iota_{e_2}\iota_{e_1} F, e_3)_- + \iota_{e_3}\iota_{(\iota_{e_2}\iota_{e_1} F)} F + \text{c.p.} \\ & = 0. \end{aligned} \tag{5.75}$$

If F is a totally anti-symmetric tensor $H_{\mu\nu\rho}\tilde{\partial}^\mu \wedge \tilde{\partial}^\nu \wedge \tilde{\partial}^\rho$, one can show that the left-hand side of (5.75) becomes $\iota_{X_3}\iota_{X_2}\iota_{X_1}\mathrm{d}H$. Therefore when $\mathrm{d}H = 0$, Axiom C1 holds and

$(\mathcal{V}, [\cdot,\cdot]_{\mathrm{F}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$ becomes the H-twisted standard Courant algebroid known in the literature. When we consider the condition $\tilde{\partial}^\mu \Phi = 0$, then as described in Section 4.1, this means that we restrict the doubled space to a leaf in the foliation $\mathcal{F}$ of $\mathcal{M}$. This leaf is interpreted as the physical space-time and $\mathrm{d}H = 0$ is nothing but the Bianchi identity for the field strength of the NS-NS B-field in type II supergravity.

On the other hand, when we consider an alternative frame $\rho_L = 0$, $\partial_\mu \Psi = 0$, we have $\mathrm{d}f = 0$, $\mathcal{L}_{X_i} = 0$ and $[\cdot,\cdot]_L = 0$. In this case, $F = R^{\mu\nu\rho}\partial_\mu \wedge \partial_\nu \wedge \partial_\rho$ is allowed and $\mathrm{d}_* R = 0$ appears as a condition. The other possibilities including $F = f_{\mu\nu}{}^\rho \tilde{\partial}^\mu \wedge \tilde{\partial}^\nu \wedge \partial_\rho$ and $F = Q_\mu{}^{\nu\rho} \tilde{\partial}^\mu \wedge \partial_\nu \wedge \partial_\rho$, would be allowed for general ρ_{V} (5.42). In particular, the role of the non-diagonal component $\tilde{\rho}^{\mu\nu}$ in (5.42) and the tensors $f_{\mu\nu}{}^\rho$, $Q_\mu{}^{\nu\rho}$ is discussed in [116].

In this Chapter, we consider the doubled structure of Vaisman algebroid on para-Hermitian manifold. We show that the failure of the derivation condition is resolved by imposing the strong constraint. With these results at hand, we found an algebraic origin ofthe strong constraint. Namely, it is an efficient condition for the derivation condition that ensures that $(L, \tilde{L})$ becomes a Lie bialgebroid.

In the latter part of this Chapter, we found that the consistency conditions of the doubled structures for ante- and pre- Courant algebroids are just the relaxed versions of the strong constraint in DFT. We also studied the consistency conditions for the twisted DFT algebroids. The twist is introduced by a $(2,1)$-tensor in the para-Hermitian manifold. We clarify that the tensor should satisfy appropriate conditions related by the anchors in the Lie algebroid pairs of the doubled structure. We showed that the (relaxed versions of) the strong constraint implies the induced Poisson structure in general. This means that the Poisson structure is closely related to the doubled nature of algebroids. Even though, this becomes trivial in the flat para-Hermitian manifold, it still provides non-trivial structures in the general curved para-Hermitian manifold.

6 Conclusions and Outlook

In this thesis, we focused on various algebroids and explore the theoretical structure of DFT through their doubled structures. Double Field Theory was a field theory with T-duality as symmetry. DFT was built from the discussion of closed String Field Theory (SFT) [64, 65]. The most basic setup, type II DFT, was defined on a doubled spacetime in $2D$ dimensions. The actions of the DFT were not only invariant to the T-duality group $O(D, D)$, but also to the generalized diffeomorphism (generalized Lie derivative). The generalized Lie derivative was an extension of the Lie derivative in Riemannian geometry, which is defined on doubled spacetime. DFT had the gauge symmetry which is governed by the C-bracket. This bracket does not satisfy the Jacobi identity. DFT also had the strong constraint for consistency. This constraint had no physical origin, it is only the closure constraint of the gauge algebra by the C-bracket.

Mathematically, the C-bracket defines Vaisman's metric algebroid (Vaisman algebroid) [72]. We focused on the algebroid and consider the mathematical origin of the strong constraint. The key was the Drinfel'd double. The Drinfel'd double of Lie algebra was well known [40]. We generalized the operation for Lie algebroids [102] and show the doubled structure of the Vaisman algebroid. In addition, we also considered the doubled structure of other algebroids described by the C-bracket.

DFT geometry was discussed in relation to the para-Hermitian geometry. We realizeed the Vaisman algebroid with the doubled structure on this geometry. We calculated the compatibility condition called the derivation condition in Chapter 3 and showed the mathematical origin of the strong constraint. We also realized the doubled structure of other algebroids described by the C-bracket, and discovered that the relaxed version of the strong constraint becomes apparent as the compatibility condition.

Summary of each chapter

Chapter 1

Chapter 1 briefly introduced the central concepts of this thesis through an introduction to string theory. There were five equivalent theories in string theory, which are related by string duality. Among them, T-duality was a duality unique to string theory that arises when a string winds around a compactified space. Duality was not usually treated in an explicit theory, since it is essentially a relation between different theories. We introduced generalized geometry and the doubled geometry as a framework to treat T-duality explicitly. In addition, We discussed Doubled Field Theory which is a field theory with T-duality as the symmetry.

Chapter 2

In Chapter 2, a very brief review of Doubled Field Theory was presented. DFT was a field theory defined on the doubled spacetime $(x, \tilde{x})$ with the winding coordinate $\tilde{x}$ added in addition to the usual momentum coordinate x in order to make T-duality explicit. DFT was a field theory defined on doubled spacetime $(x, \tilde{x})$. DFT was historically a theory constructed from SFT, but it can also be interpreted as a recombination of supergravity into a T-duality covariant. The most basic DFT was based on type II theory. The type II DFT action was not only invariant under the T-duality group $O(D, D)$, but also under the generalized Lie derivative. The DFT has gauge symmetry, which was described by the C-bracket. The C-bracket was characterized by the fact that it does not satisfy the Jacobi identity. This was because the generalized diffeomorphism in DFT is a combination of the usual spacetime diffeomorphism and the gauge transformation of the B field. For consistency of the theory, DFT had the strong constraint. This condition had no physical origin and is a condition for the closure of the Jacobi identity by the C-bracket. Preparations for finding the algebraic origin of this condition were made in the next Chapter 3 and Chapter 4.

Chapter 3

In Chapter 3, we summarized algebroids and their doubled structures, which are related to gauge symmetry in DFT. The basic idea of this chapter was the Drinfel'd double for Lie algebra. This was the operation of introducing two Lie algebras in a dual vector space and taking their direct sum. If the compatibility condition held between Lie algebras, the structure after the direct summation also forms a new Lie algebra. Based on this relationship, we first introduced Lie algebroid, the simplest algebroid structure. Lie algebroid was an extension of the Lie algebra structure to vector bundles on a manifold M. By preparing a dual vector bundle and introducing two Lie algebroids, Drinfel'd double could be performed on those Lie algebroids under the compatibility condition. This compatibility condition was called the derivation condition. However, it was not the Lie algebroid but the Courant algebroid that appears in Drinfel'd double of the Lie bialgebroids. The Courant algebroid was a structure defined by five axioms including the (deformed) the Jacobi identity. The Vaisman algebroid was a further generalization of the Courant algebroid, satisfying only two axioms. Just as the Courant algebroid was obtained by Drinfel'd double of Lie algebroid, the Vaisman algebroid was obtained by Drinfel'd double of Lie algebroid. The Vaisman algebroid also proved to have a doubled structure [85].

We also classified all algebroids that could be defined by the C-bracket. Some of the five axioms defining Courant algebroids were not independent of each other. By paying attention to the principal-subordinate relationship of axioms and examining all possible combinations of axioms, we could find four new algebroid structures in addition to the Courant and the Vaisman algebroids. Furthermore, we were able to show that two of the new algebroids have doubled structures. Finally, we discussed the relationship between these algebroids and algebras.

Chapter 4

In Chapter 4, we summarized the geometries related to T-duality. First, generalized geometry dealt with generalized tangent bundles. The generalized tangent bundle $\mathbb{T}M$ was a direct sum of the tangent bundle TM and the cotangent bundle T^*M. Since the symmetry group of the target space was $O(D,D)$ and the gauge transformation of the B-field can be taken in geometrically, this geometry is especially compatible with supergravity. The generalized Lie derivative of the generalized vector $\Gamma(\mathbb{T}M)$ was related to the Courant algebroid mentioned in Chapter 3.

However, the geometry of DFT was not generalized geometry but the doubled geometry. Since the base space (spacetime) itself was doubled, the fundamental idea was different from that of generalized geometry. The doubled geometry was related to the para-Hermitian geometry and the Born geometry. First, the necessary structures for the para-Hermitian geometry were defined for a $2D$ manifold which is called the para-complex structure. Next, we considered the split of tangent bundle $T\mathcal{M}$ to L and $\tilde{L}$ and disscused the integrability of $L, \tilde{L}$. We showed that the doubled spacetime on $\mathcal{M}$ could be interpreted as a leaf of foliation. We also introduced natural isomorphism and discussed the relationship between the para-Hermitian geometry and generalized geometry. Next, we introduced the Born geometry, a generalization of the para-Hermitian geometry. Born geometry had three internal structures, which form a para-quaternionic structure by compatibility.

Chapter 5

Continuing from Chapter 5, we reproduced the concepts appearing in DFT within the framework of the para-Hermitian geometry. In Chapter 5, we reproduced the doubled structure of the Vaisman algebroid discussed in Chapter 2 on the para-Hermitian manifold $\mathcal{M}$. First, we introduced the para-Dolbeault operators. We specified the exterior algebras for $L, \tilde{L}$ and constructed a para-Dolbeault cohomology.

From component calculations, the differential operator $\tilde{\mathrm{d}}$ and the Lie bracket $[\cdot,\cdot]_L$ did not satisfy the derivation condition for $L, \tilde{L}$. Thus, the pair $(L, \tilde{L})$ was not a Lie bialgebroid in general from the discussion in Chapter 3. We realized the Vaisman algebroid defined by the C-bracket on the para-Hermitian manifold. We also showed that the algebraic origin of the strong constraint condition of DFT was the derivation condition. In addition, we reproduced the structure of the new algebroids with doubled structure in the same way as presented in Chapter 2. As a result, we found that the conditions for constructing these new algebroids were a relaxation of certain strong constraints.

In this thesis, we paid particular attention to the gauge algebra (algebroid) of DFT. The results were very interesting because it is clear that there were various mathematical structures from DFT. These structures had not appeared in conventional physics. There were still many other topics related to the algebroids and the doubled geometry.

Outlook

Finally, we would like to introduce some topics not covered in this thesis. As already mentioned, in DFT, the C-bracket is characterized by its gauge symmetry. In other words, this is an infinitesimal gauge transformation. In contrast, finite gauge transformations have also been discussed in [127, 139–145]. In particular, from a mathematical point of view, finite gauge transformation of the DFT is discussed by an "integration" operation on the Vaisman algebroid. The "integration" means an analogy with the fact that Lie groups can be constructed from Lie algebras. At least, it has already been shown that the Poisson Lie group can be obtained by integration of Lie bialgebras, and its generalization to Poisson Lie groupoids by integration of Lie bialgebroid [108, 146]. A groupoid is an extended structure of the group such that it has more than one unit element. For details, see [147]. Although integrals of algebroids are generally difficult, there is a prediction that a groupoid structure becomes apparent by integration of the Courant algebroids [148,149]. As a further argument, there is a paper that points out the connection between the integral of the Vaisman algebroid and Lackoid [150]. The problem of obtaining a group (or groupoid) from an algebra (or algebroid) by integration is called the *coquecigrue problem*. The coquecigrue refers to an imaginary creature that appears in the 16th-century French literature of the Gargantua stories and in the Pantagruel. If the coquecigrue problem for the Vaisman algebroid is solved, it is not only a mathematically interesting result, but also it would give a geometrical origin for the gauge symmetry of the DFT.

Moreover, imposing strong constraints is a sufficient condition for obtaining a closed gauge algebra of the DFT, not a necessary condition. More generally, a DFT setup that does not require strong constraints is also considered [83]. The pre-Courant algebroid and the ante-Courant algebroid may be related to relaxing the strong constraint.

In this thesis, we introduce the para-Hermitian geometry and the Born geometry to realize the doubled spacetime. Furthermore, other method using graded geometry are also considered. The relationship of the Courant algebroids and QP manifolds are discussed in [151]. The relationship between the structure of the C-bracket and the derived bracket product [152, 153] should also sort out the relationship between graded geometry and the para-Hermitian geometry.

The relationship with Poisson-Lie T-duality [118, 154, 155], an extension of T-duality, is also discussed. When the DFT is constructed on group manifolds based on type II supergravity theory, Poisson-Lie T-duality is an obvious symmetry of the theory. Since the Vaisman algebroid with the doubled structure constructs on any base space (not only the para-Hermitian manifold), we would like to see what kind of results we can obtain when we reproduce it on a group manifold. it is said that the Courant algebroid by Drinfel'd double of [102] is the key to understanding Poisson-Lie T-duality in string theory [156–159].

Finally, a naive extension of DFT itself is to upgrade the symmetry from T-duality to U-duality. Exceptional Field Theory (EFT) was invented to make U-duality an explicit symmetry [160]. EFT is also defined on extended spacetime. For this reason, the recent development of EFT and (type II and heterotic) DFT are now collectively called Extended Field Theory (ExFT). The geometry of ExFT is also called Extended Geometry. The gauge symmetry of EFT has been discussed in [161] and others, but its geometric picture is more unclear than that of DFT. In a naive view, since EFT is an extension of DFT, the geometric picture of EFT should encompass doubling geometry. It is not well known how to geometrically treat the space in the gauge direction in heterotic DFT. Questions remain as to how the gauge symmetry of ExFT is related to the doubled structure treated in this study.

A Calculation Notes

A.1 Derive the formula (3.29)

We derive the following equation.

$$\begin{aligned} T(e_1, e_2, e_3) &\equiv \frac{1}{3}(([e_1, e_2]_{\mathrm{V}}, e_3)_+ + \text{c.p.}) \\ &= \frac{1}{2}\langle \xi_3, [X_1, X_2]_E \rangle + \langle [\xi_1, \xi_2]_{E^*}, X_3 + \rho(X_3)(e_1, e_2)_- - \rho_*(\xi_3)(e_1, e_2)_- \rangle \end{aligned} \tag{3.29}$$

This is the relation corresponding to Lemma 3.2 of Ref. [102], which can be shown by calcration about $T(e_1, e_2, e_3)$. Since the form of the bracket product is the same, Using the Vaisman bracket $[\cdot, \cdot]_{\mathrm{V}}$, or with the Courant bracket $[\cdot, \cdot]_{\mathrm{c}}$ in [102], the same relational expression is obtained.

First, focusing on $([e_1, e_2]_{\mathrm{V}}, e_3)_+$, from the definition of $[\cdot, \cdot]_{\mathrm{V}}, (\cdot, \cdot)_+$,

$$\begin{aligned} ([e_1, e_2]_{\mathrm{V}}, e_3)_+ = \frac{1}{2}\{ &\langle \xi_3, [X_1, X_2]_E \rangle + \langle \xi_3, \mathcal{L}_{\xi_1} X_2 \rangle - \langle \xi_3, \mathcal{L}_{\xi_2} X_1 \rangle - \rho_*(\xi_3)(e_1, e_2)_- \\ &\langle [\xi_1, \xi_2]_{E^*}, X_3 \rangle + \langle \mathcal{L}_{X_1}\xi_2, X_3 \rangle - \langle \mathcal{L}_{X_2}\xi_1, X_3 \rangle + \rho(X_3)(e_1, e_2)_-, \end{aligned} \tag{A.1}$$

Focusing on the terms of the Lie derivative, the following relations can be used from the distributive property of the Lie derivative

$$\begin{aligned} \langle \xi_3, \mathcal{L}_{\xi_1} X_2 \rangle &= \mathcal{L}_{\xi_1}\langle \xi_3, X_2 \rangle - \langle [\xi_1, \xi_3]_{E^*}, X_2 \rangle \\ \langle \mathcal{L}_{X_1}\xi_2, X_3 \rangle &= \mathcal{L}_{X_1}\langle \xi_2, X_3 \rangle - \langle \xi_2, [X_1, X_3]_E \rangle \end{aligned}$$

Also, from the definition of d in [108],

$$\mathcal{L}_{X_1}\langle \xi_2, X_3 \rangle = \iota_{X_1}\mathrm{d}\langle \xi_2, X_3 \rangle = \rho(X_1)\langle \xi_2, X_3 \rangle. \tag{A.2}$$

Similarly,

$$\mathcal{L}_{\xi_1}\langle \xi_3, X_2 \rangle = \rho_*(\xi_1)\langle \xi_3, X_2 \rangle. \tag{A.3}$$

Therefore, $([e_1, e_2]_{\mathrm{V}}, e_3)_+$can be further rewritten from (A.1) as follows.

$$\begin{aligned} ([e_1, e_2]_{\mathrm{V}}, e_3)_+ = &\frac{1}{2}\{\langle \xi_3, [X_1, X_2]_E \rangle + \langle [\xi_1, \xi_2]_{E^*}, X_3 \rangle + \text{c.p.}\} \\ &+ \frac{1}{2}\{\rho_*(\xi_1)\langle \xi_3, X_2 \rangle - \rho_*(\xi_2)\langle \xi_3, X_1 \rangle - \rho_*(\xi_3)(e_1, e_2)_- \\ &\quad + \rho(X_1)\langle \xi_2, X_3 \rangle - \rho(X_2)\langle \xi_1, X_3 \rangle + \rho(X_3)(e_1, e_2)_-\} \end{aligned} \tag{A.4}$$

Next, we add and subtract$\rho(X_1)(e_2,e_3)_-$, $\rho(X_2)(e_3,e_1)_-$ and $\rho_*(\xi_1)(e_2,e_3)$, $\rho_*(\xi_2)(e_3,e_1)_-$ respectively, and organise the cyclic terms. From the definition of ρ, $([e_1,e_2]_{\rm V},e_3)_+$ becomes the following form.

$$\begin{aligned}([e_1,e_2]_{\rm V},e_3)_+ &= \frac{1}{2}\{\langle\xi_3,[X_1,X_2]_E\rangle + \langle X_3,[\xi_1,\xi_2]_{E^*}\rangle + \rho(X_3)(e_1,e_2)_- - \rho_*(\xi_3)(e_1,e_2)_-\text{c.p.}\}\\ &\quad + \frac{1}{2}\rho(e_1)(e_2,e_3)_+ - \frac{1}{2}\rho(e_2)(e_3,e_1)_+ \end{aligned} \tag{A.5}$$

Now, if we cycle through the legs of $([e_1,e_2]_{\rm V},e_3)_+$ and take the sum, we get the right-hand side of (3.29).

$$\begin{aligned}T(e_1,e_2,e_3) &\equiv \frac{1}{3}(([e_1,e_2]_{\rm V},e_3)_+ + \text{c.p.})\\ &= \frac{1}{2}\{\langle[X_1,X_2]_E,\xi_3\rangle + \langle[\xi_1,\xi_2]_{E^*},X_3\rangle + \rho(X_3)(e_1,e_2)_- - \rho_*(\xi_3)(e_1,e_2)_-\text{c.p.}\}\\ &\quad + \frac{1}{2}\{\rho(e_1)(e_2,e_3)_+ - \rho(e_2)(e_3,e_1)_+ + \rho(e_2)(e_3,e_1)_+\\ &\qquad - \rho(e_3)(e_1,e_2)_+ + \rho(e_3)(e_1,e_2)_+ - \rho(e_1)(e_2,e_3)_+\}\\ &= \frac{1}{2}\{\langle\xi_3,[X_1,X_2]_E\rangle + \langle[\xi_1,\xi_2]_{E^*},X_3\rangle\\ &\qquad + \rho(X_3)(e_1,e_2)_- - \rho_*(\xi_3)(e_1,e_2)_- + \text{c.p.}\}\end{aligned} \tag{A.6}$$

Therefore, it can be shown that (3.29).

A.2 Derive the formula (3.30)

We derive the following equation.

$$\begin{aligned}([e_1,e_2]_{\rm V},e_3)_- + \text{c.p.} &= T(e_1,e_2,e_3)\\ &\quad + [\{\rho(X_3)(e_1,e_2)_- + 2\rho_*(\xi_3)(e_1,e_2)_- - \langle[\xi_1,\xi_2]_{E^*},X_3\rangle\} + \text{c.p.}]\end{aligned} \tag{3.30}$$

This is a relation corresponding to Lemma 3.4 of [102] and can be shown from the definition of $(\cdot,\cdot)_\pm$.

From the definition of $(\cdot,\cdot)_\pm$, the following relation is obtained.

$$\begin{aligned}([e_1,e_2]_{\rm V},e_3)_- + ([e_1,e_2]_{\rm V},e_3)_+ &= \langle[\xi_1,\xi_2]_{E^*},X_3\rangle + \langle\mathcal{L}_{X_1}\xi_2,X_3\rangle\\ &\quad - \langle\mathcal{L}_{X_2}\xi_1,X_3\rangle + \langle \mathrm{d}(e_1,e_2)_-,X_3\rangle.\end{aligned} \tag{A.7}$$

Furthermore, organising the right-hand side,

$$\begin{aligned}([e_1,e_2]_{\rm V},e_3)_- + ([e_1,e_2]_{\rm V},e_3)_+ &= \langle[\xi_1,\xi_2]_{E^*},X_3\rangle + \rho(X_1)\langle\xi_2,X_3\rangle - \langle\xi_2,[X_1,X_3]_E\rangle\\ &\quad - \rho(X_2)\langle\xi_1,X_3\rangle + \langle\xi_1,[X_2,X_3]_E\rangle + \rho(X_3)(e_1,e_2)_-.\end{aligned} \tag{A.8}$$

We sum the index of (A.8) and use (3.29) to obtain following result,

$$\begin{aligned}
&\{([e_1, e_2]_{\mathrm{V}}, e_3)_- + ([e_1, e_2]_{\mathrm{V}}, e_3)_+\} + \text{c.p.} \\
&= \{([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.}\} + 3T(e_1, e_2, e_3) \\
&= \{\langle [\xi_1, \xi_2]_{E^*}, X_3\rangle + \rho(X_1)\langle \xi_2, X_3\rangle - \langle \xi_2, [X_1, X_3]_E\rangle \\
&\quad - \rho(X_2)\langle \xi_1, X_3\rangle + \langle \xi_1, [X_2, X_3]_E\rangle + \rho(X_3)(e_1, e_2)_-\} + \text{c.p.} \qquad (\text{A.9})
\end{aligned}$$

Once we calculate all the c.p. (cyclic permutarion) parts as well, it can be summarised as follows

$$\begin{aligned}
&\{([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.}\} + 3T(e_1, e_2, e_3) \\
&= \{\langle [\xi_1, \xi_2]_{E^*}, X_3\rangle + 2\langle \xi_3, [X_1, X_2]_E\rangle + 3\rho(X_3)(e_1, e_2)_-\} + \text{c.p.} \\
&= 4T(e_1, e_2, e_3) + [\{\rho(X_3)(e_1, e_2)_- + 2\rho_*(\xi_3)(e_1, e_2)_- - \langle [\xi_1, \xi_2]_{E^*}, X_3\rangle\} + \text{c.p.}] \qquad (\text{A.10})
\end{aligned}$$

The transformation to the last line is just a forced creation of $4T(e_1, e_2, e_3)$ to land on (3.30). If we migrate and organise (A.10), we get (3.30).

A.3 Axiom C1

Calculate the left-hand side of Axiom C1 $[[e_1, e_2]_{\mathrm{V}}, e_3]_{\mathrm{V}} + \text{c.p.}$ From the definition of $[\cdot, \cdot]_{\mathrm{V}}$, it follows that

$$[[e_1, e_2]_{\mathrm{V}}, e_3]_{\mathrm{V}} + \text{c.p.} = I_1 + I_2, \qquad (\text{A.11})$$

$$\begin{aligned}
I_1 = {} & [[\xi_1, \xi_2]_{E^*}, \xi_3]_{E^*} + [\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1, \xi_3]_{E^*} + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} \\
& + \mathcal{L}_{[X_1, X_2]_E + \mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1 - \mathrm{d}_*(e_1, e_2)_-}\xi_3 \\
& - \mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{X_3}\mathcal{L}_{X_1}\xi_2 + \mathcal{L}_{X_3}\mathcal{L}_{X_2}\xi_1 \\
& - \mathcal{L}_{X_3}\mathrm{d}(e_1, e_2)_- + \mathrm{d}([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.}, \qquad (\text{A.12})
\end{aligned}$$

$$\begin{aligned}
I_2 = {} & [[X_1, X_2]_E, X_3]_E + [\mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1, X_3]_E - [\mathrm{d}_*(e_1, e_2)_-, X_3]_E \\
& + \mathcal{L}_{[\xi_1, \xi_2]_{E^*} + \mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1 + \mathrm{d}(e_1, e_2)_-}X_3 \\
& - \mathcal{L}_{\xi_3}[X_1, X_2]_E - \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_1}X_2 + \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_2}X_1 \\
& + \mathcal{L}_{\xi_3}\mathrm{d}_*(e_1, e_2)_- - \mathrm{d}_*([e_1, e_2]_{\mathrm{V}}, e_3)_- + \text{c.p.} \qquad (\text{A.13})
\end{aligned}$$

Let $\Gamma(E^*)$ component be I_1 and $\Gamma(E)$ component be I_2. Since I_1 and I_2 are calculated almost identically, only I_1 is taken out and calculated. Using

$$\mathcal{L}_{[X_1, X_2]_E} = [\mathcal{L}_{X_1}, \mathcal{L}_{X_2}]_E \qquad (\text{A.14})$$

we obtain

$$\mathcal{L}_{[X_1, X_2]_E}\xi_3 - \mathcal{L}_{X_3}\mathcal{L}_{X_1}\xi_2 + \mathcal{L}_{X_3}\mathcal{L}_{X_2}\xi_1 + \text{c.p.} = 0. \qquad (\text{A.15})$$

So I_1 can be written as follows.

$$\begin{aligned}
I_1 &= \{[[\xi_1,\xi_2]_{E^*},\xi_3]_{E^*} + [\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1,\xi_3]_{E^*} - [\mathrm{d}_*(e_1,e_2)_-,\xi_3]_{E^*}\\
&\quad + \mathcal{L}_{[X_1,X_2]_E+\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1-\mathrm{d}_*(e_1,e_2)_-}\xi_3\\
&\quad - \mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} - \mathcal{L}_{X_3}\mathcal{L}_{X_1}\xi_2 + \mathcal{L}_{X_3}\mathcal{L}_{X_2}\xi_1 - \mathcal{L}_{\mathrm{d}(e_1,e_2)_-} + \mathrm{d}([e_1,e_2]_{\mathrm{V}},e_3)_-\} + \text{c.p.}\\
&= \{[\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1,\xi_3]_{E^*} - [\mathrm{d}_*(e_1,e_2)_-,\xi_3]_{E^*}\\
&\quad + \mathcal{L}_{\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1-\mathrm{d}_*(e_1,e_2)_-}\xi_3\\
&\quad - \mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} - \mathcal{L}_{\mathrm{d}(e_1,e_2)_-} + \mathrm{d}([e_1,e_2]_{\mathrm{V}},e_3)_-\} + \text{c.p.}
\end{aligned} \tag{A.16}$$

We now turn our attention to the term $\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} + \text{c.p.}$.

$$\begin{aligned}
\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} &= (\mathrm{d}\iota_{X_3} + \iota_{X_3}\mathrm{d})[\xi_1,\xi_2]_{E^*}\\
&= \mathrm{d}\langle X_3,[\xi_1,\xi_2]_{E^*}\rangle + i_{X_3}\mathrm{d}[\xi_1,\xi_2]_{E^*}\\
&\quad + (\iota_{X_3}\mathcal{L}_{\xi_1}\mathrm{d}\xi_2 - \iota_{X_3}\mathcal{L}_{\xi_1}\mathrm{d}\xi_2) + (\iota_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1 - \iota_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1)\\
&= \mathrm{d}\langle X_3,[\xi_1,\xi_2]_{E^*}\rangle + \iota_{X_3}\mathcal{L}_{\xi_1}\mathrm{d}\xi_2 - \iota_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1\\
&\quad + \iota_{X_3}(\mathrm{d}[\xi_1,\xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1).
\end{aligned} \tag{A.17}$$

Furthermore, when $X \in \Gamma(E),\ \xi,\eta \in \Gamma(E^*)$, the following command holds for $\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*}$ in general.

$$\iota_X\mathcal{L}_\xi\mathrm{d}\eta = [\xi,\mathcal{L}_X\eta]_{E^*} - \mathcal{L}_{\mathcal{L}_\xi X}\eta + [\mathrm{d}\langle\eta,X\rangle,\xi]_{E^*} + \mathrm{d}(\rho_*(\xi)\langle\eta,X\rangle) - \mathrm{d}\langle[\xi,\eta]_{E^*},X\rangle \tag{A.18}$$

The derivation of (A.18) is given in the next section. Applying this to $i_{X_3}\mathcal{L}_{\xi_1}\mathrm{d}\xi_2$ and $i_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1$, $\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} + \text{c.p.}$ takes the following form.

$$\begin{aligned}
\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} + c.p &= \{[\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1,\xi_3]_{E^*} + \mathcal{L}_{\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1}\xi_3\\
&\quad + 2[\mathrm{d}(e_1,e_2)_-,\xi_3]_{E^*} + 2\mathrm{d}(\rho_*(\xi_3)(e_1,e_2)_-) - \mathrm{d}\langle[\xi_1,\xi_2]_{E^*},X_3\rangle\\
&\quad + \iota_{X_3}(\mathrm{d}[\xi_1,\xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1)\} + \text{c.p.}
\end{aligned} \tag{A.19}$$

The derivation of (A.19) is given in the next following section.

Substituting (A.19) into (A.16), I_1 can be further organised and

$$\begin{aligned}
I_1 &= \{\mathrm{d}\{([e_1,e_2]_{\mathrm{V}},e_3)_- - \rho(X_3)(e_1,e_2)_- - 2\rho_*(\xi_3)(e_1,e_2)_- + \langle[\xi_1,\xi_2]_{E^*},X_3\rangle - K_1 - K_2\} + \text{c.p.},\\
K_1 &= \iota_{X_3}(\mathrm{d}[\xi_1,\xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1),\\
K_2 &= \mathcal{L}_{\mathrm{d}(e_1,e_2)_-}\xi_3 + [\mathrm{d}(e_1,e_2)_-,\xi_3]_{E^*}.
\end{aligned} \tag{A.20}$$

From (3.30), using

$$\begin{aligned}
([e_1,e_2]_{\mathrm{V}},e_3)_- + \text{c.p.} &= T(e_1,e_2,e_3)\\
&\quad + [\{\rho(X_3)(e_1,e_2)_- + 2\rho_*(\xi_3)(e_1,e_2)_- - \langle[\xi_1,\xi_2]_{E^*},X_3\rangle + \text{c.p.}\}],
\end{aligned} \tag{3.30}$$

I_1 has the following form.

$$I_1 = \mathrm{d}T(e_1, e_2, e_3) - \{K_1 + K_2\} + \text{c.p.}, \tag{A.21}$$

here,

$$\begin{aligned} K_1 &= \iota_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1), \\ K_2 &= \mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}\xi_3 + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*}. \end{aligned} \tag{A.22}$$

The same applies for I_2.

$$\begin{aligned} I_2 &= \mathrm{d}_*T(e_1, e_2, e_3) - \{K_3 + K_4\} + \text{c.p.}, \\ K_3 &= \iota_{\xi_3}(\mathrm{d}_*[X_1, X_2]_E - \mathcal{L}_{X_1}\mathrm{d}_*X_2 + \mathcal{L}_{X_2}\mathrm{d}_*X_1), \\ K_4 &= \mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}X_3 + [\mathrm{d}_*(e_1, e_2)_-, X_3]_E. \end{aligned} \tag{A.23}$$

Thus, the final result of the calculation of $[[e_1, e_2]_{\mathrm{V}}, e_3]_{\mathrm{V}} + c.p$ is obtained as follows.

$$\begin{aligned} [[e_1, e_2]_{\mathrm{V}}, e_3]_{\mathrm{V}} + c.p &= I_1 + I_2 \\ &= \mathcal{D}T(e_1, e_2, e_3) - (J_1 + J_2 + \text{c.p.}). \end{aligned} \tag{A.24}$$

Here,

$$\begin{aligned} J_1 &= K_1 + K_3 \\ &= \iota_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1) + \iota_{\xi_3}(\mathrm{d}_*[X_1, X_2]_E - \mathcal{L}_{X_1}\mathrm{d}_*X_2 + \mathcal{L}_{X_2}\mathrm{d}_*X_1), \\ J_2 &= K_2 + K_4 \\ &= \mathcal{L}_{\mathrm{d}(e_1,e_2)_-}\xi_3 + [\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} + \mathcal{L}_{\mathrm{d}_*(e_1,e_2)_-}X_3 + [\mathrm{d}_*(e_1, e_2)_-, X_3]_E. \end{aligned} \tag{A.25}$$

In general, for any X_i, ξ_i, f, $(J_1 + J_2 + \text{c.p.})$is not 0. Therefore, Axiom C1 is broken in $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$.

A.3.1 Proof of formula (A.18)

We proof

$$\iota_X\mathcal{L}_\xi\mathrm{d}\eta = [\xi, \mathcal{L}_X\eta]_{E^*} - \mathcal{L}_{\mathcal{L}_\xi X}\eta + [\mathrm{d}\langle\eta, X\rangle, \xi]_{E^*} + \mathrm{d}(\rho_*(\xi)\langle\eta, X\rangle) - \mathrm{d}\langle[\xi, \eta]_{E^*}, X\rangle. \quad ((\text{A.18}))$$

As it is difficult to calculate the left-hand side directly, we calculate $\langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta, Y\rangle$ and check that the right-hand side and Y form an inner product, as follows.

$$\begin{aligned} \langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta, Y\rangle = {}& \langle[\xi, \mathcal{L}_X\eta]_{E^*}, Y\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta, Y\rangle - \langle[\xi, \mathrm{d}\langle\eta, X\rangle]_{E^*}, Y\rangle \\ & + \langle\mathrm{d}(\rho_*(\xi)\langle\eta, X\rangle), Y\rangle + \langle\mathrm{d}\langle[\xi, \eta]_{E^*}, X\rangle, Y\rangle \end{aligned} \tag{A.26}$$

First, from the distributive law of the Lie derivative, it follows that

$$\langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta, Y\rangle = \mathcal{L}_\xi(\mathrm{d}\eta(X, Y)) - \mathrm{d}\eta(\mathcal{L}_\xi X, Y) - \mathrm{d}\eta(X, \mathcal{L}_\xi Y). \tag{A.27}$$

The first term of the right-hand side can be replaced by $\rho_*(\xi)\mathrm{d}\eta(X,Y)$ by using (A.3). Also, from the definition of the outer differential operator, we have

$$\mathrm{d}\xi(X,Y) = \rho(X)\langle\xi,Y\rangle - \rho(Y)\langle\xi,X\rangle - \langle\xi,[X,Y]_E\rangle. \tag{A.28}$$

Applying this to each term, (A.27) can be transformed as follows.

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta,Y\rangle = {} & \rho_*(\xi)\rho(X)\langle\xi,Y\rangle - \rho_*(\xi)\rho(Y)\langle\eta,X\rangle - \rho_*(\xi)\langle\eta,[X,Y]_E\rangle \\ & - \rho(\mathcal{L}_\xi X)\langle\eta,Y\rangle + \rho(Y)\langle\eta,\mathcal{L}_\xi X\rangle + \langle\eta,[\mathcal{L}_\xi X,Y]_E\rangle \\ & + \rho(\mathcal{L}_\xi Y)\langle\eta,X\rangle - \rho(X)\langle\eta,\mathcal{L}_\xi Y\rangle - \langle\eta,[\mathcal{L}_\xi Y,X]_E\rangle.\end{aligned} \tag{A.29}$$

Furthermore, using (A.2),(A.3)and (3.17) the first term in each row of (A.29) can be expansion as follows.

$$\rho_*(\xi)\rho(X)\langle\xi,Y\rangle = \rho_*(\xi)\langle\mathcal{L}_X\eta,Y\rangle + \rho_*(\xi)\langle\eta,[X,Y]_E\rangle \tag{A.30}$$

$$\rho(\mathcal{L}_\xi X)\langle\eta,Y\rangle = \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta,Y\rangle + \langle\eta,[\mathcal{L}_\xi X,Y]_E\rangle \tag{A.31}$$

$$\rho(\mathcal{L}_\xi Y)\langle\eta,X\rangle = \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle + \langle\eta,[\mathcal{L}_\xi Y,X]_E\rangle \tag{A.32}$$

Substituting and organising this, some of the terms cancel each other out and coalesce as follows.

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta,Y\rangle = {} & \rho_*(\xi)\langle\mathcal{L}_X\eta,Y\rangle - \rho_*(\xi)\rho(Y)\langle\eta,X\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta,Y\rangle \\ & + \rho(Y)\langle\eta,\mathcal{L}_\xi X\rangle - \rho(X)\langle\eta,\mathcal{L}_\xi Y\rangle + \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle.\end{aligned} \tag{A.33}$$

we rewrite this result in the form of an inner product with Y without using an anchor. First, we focus on $\rho(Y)\langle\eta,\mathcal{L}_\xi X\rangle$. This term can be rewritten from (A.2),(A.3) as

$$\rho(Y)\langle\eta,\mathcal{L}_\xi X\rangle = \langle\mathrm{d}(\rho_*(\xi)\langle\eta,X\rangle),Y\rangle - \langle\mathrm{d}\langle[\xi,\eta]_{\mathrm{V}},X\rangle,Y\rangle. \tag{A.34}$$

We substitute this,

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta,Y\rangle = {} & \langle\mathrm{d}(\rho_*(\xi)\langle\eta,X\rangle),Y\rangle + \langle\mathrm{d}\langle[\xi,\eta]_{\mathrm{V}},X\rangle,Y\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta,Y\rangle \\ & + \rho_*(\xi)\langle\mathcal{L}_X\eta,Y\rangle - \rho_*(\xi)\rho(Y)\langle\eta,X\rangle - \rho(X)\langle\eta,\mathcal{L}_\xi Y\rangle + \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle.\end{aligned} \tag{A.35}$$

Similarly, using (A.2),(A.3)to organise the second stage of (A.35), we obtain the following.

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi\mathrm{d}\eta,Y\rangle = {} & \langle\mathrm{d}(\rho_*(\xi)\langle\eta,X\rangle),Y\rangle + \langle\mathrm{d}\langle[\xi,\eta]_{\mathrm{V}},X\rangle,Y\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta,Y\rangle \\ & + \langle[\xi,\mathcal{L}_X\eta]_{\mathrm{V}},Y\rangle - \mathcal{L}_\xi\langle\mathcal{L}_Y\eta,X\rangle - \mathcal{L}_\xi\langle\eta,\mathcal{L}_Y X\rangle - \langle\eta,\mathcal{L}_X\mathcal{L}_\xi Y\rangle + \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle.\end{aligned} \tag{A.36}$$

Here,

$$\begin{aligned}-\langle\eta,\mathcal{L}_X\mathcal{L}_\xi Y\rangle + \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle &= \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle + \langle\mathcal{L}_{\mathcal{L}_\xi Y}\eta,X\rangle \\ &= \langle\mathrm{d}\langle\eta,X\rangle,\mathcal{L}_\xi Y\rangle.\end{aligned} \tag{A.37}$$

So,

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi \mathrm{d}\eta, Y\rangle &= \langle \mathrm{d}(\rho_*(\xi)\langle\eta, X\rangle), Y\rangle + \langle \mathrm{d}\langle[\xi,\eta]_{\mathrm{V}}, X\rangle, Y\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta, Y\rangle \\ &\quad + \langle[\xi, \mathcal{L}_X\eta]_{\mathrm{V}}, Y\rangle - \mathcal{L}_\xi\langle\mathcal{L}_Y\eta, X\rangle - \mathcal{L}_\xi\langle\eta, \mathcal{L}_Y X\rangle + \langle \mathrm{d}\langle\eta, X\rangle, \mathcal{L}_\xi Y\rangle. \end{aligned} \tag{A.38}$$

From the distributive law of the Lie derivative, it follows that

$$\langle \mathrm{d}\langle\eta, X\rangle, \mathcal{L}_\xi Y\rangle = \mathcal{L}_\xi\langle \mathrm{d}\langle\eta, X\rangle, Y\rangle - \langle[\xi, \mathrm{d}\langle\eta, X\rangle]_{\mathrm{V}}, Y\rangle. \tag{A.39}$$

therefore,

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi \mathrm{d}\eta, Y\rangle &= \langle \mathrm{d}(\rho_*(\xi)\langle\eta, X\rangle), Y\rangle + \langle \mathrm{d}\langle[\xi,\eta]_{E^*}, X\rangle, Y\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta, Y\rangle \\ &\quad + \langle[\xi, \mathcal{L}_X\eta]_{E^*}, Y\rangle - \langle[\xi, \mathrm{d}\langle\eta, X\rangle]_{E^*}, Y\rangle \\ &\quad - \mathcal{L}_\xi\langle\mathcal{L}_Y\eta, X\rangle - \mathcal{L}_\xi\langle\eta, \mathcal{L}_Y X\rangle + \mathcal{L}_\xi\langle \mathrm{d}\langle\eta, X\rangle, Y\rangle. \end{aligned} \tag{A.40}$$

Here, the third line cancels out from the following calculation.

$$\begin{aligned}\mathcal{L}_\xi\langle \mathrm{d}\langle\eta, X\rangle, Y\rangle &= \mathcal{L}_\xi\mathcal{L}_Y\langle\eta, X\rangle \\ &= \mathcal{L}_\xi\langle\mathcal{L}_Y\eta, X\rangle + \mathcal{L}_\xi\langle\eta, \mathcal{L}_Y X\rangle. \end{aligned} \tag{A.41}$$

therefore,

$$\begin{aligned}\langle\iota_X\mathcal{L}_\xi \mathrm{d}\eta, Y\rangle &= \langle \mathrm{d}(\rho_*(\xi)\langle\eta, X\rangle), Y\rangle + \langle \mathrm{d}\langle[\xi,\eta]_{E^*}, X\rangle, Y\rangle - \langle\mathcal{L}_{\mathcal{L}_\xi X}\eta, Y\rangle \\ &\quad + \langle[\xi, \mathcal{L}_X\eta]_{E^*}, Y\rangle - \langle[\xi, \mathrm{d}\langle\eta, X\rangle]_{E^*}, Y\rangle. \end{aligned} \tag{A.42}$$

A.3.2 Proof of formula (A.19)

We proof following fomula.

$$\begin{aligned}\mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*} + \text{c.p.} &= [\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1, \xi_3]_V + \mathcal{L}_{\mathcal{L}_{\xi_1}X_2 - \mathcal{L}_{\xi_2}X_1}\xi_3 \\ &\quad + 2[\mathrm{d}(e_1, e_2)_-, \xi_3]_{E^*} + 2\mathrm{d}(\rho_*(\xi_3)\cdot(e_1, e_2)_-) - \mathrm{d}\langle[\xi_1, \xi_2]_{E^*}, X_3\rangle \\ &\quad + i_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1) + \text{c.p.} \end{aligned} \tag{A.19}$$

First, after calculating $\mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*}$ on the left-hand side, the right-hand side is confirmed by taking its circular sum. The $\mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*}$ can be expanded as follows according to the definition of the Lie derivative. However, below the second equal sign, the terms in $\iota_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1$ and the term in $\mathcal{L}_{\xi_2}\mathrm{d}\xi_1$ are deliberately added and subtracted respectively. This operation allows the terms affected by the derivation condition to be made explicit.

$$\begin{aligned}\mathcal{L}_{X_3}[\xi_1, \xi_2]_{E^*} &= (\mathrm{d}\iota_{X_3} + \iota_{X_3}\mathrm{d})[\xi_1, \xi_2]_{E^*} \\ &= \mathrm{d}\langle[\xi_1, \xi_2]_{E^*}, X_3\rangle + \iota_{X_3}\mathcal{L}_{\xi_1}\mathrm{d}\xi_2 - \iota_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1 \\ &\quad + \iota_{X_3}(\mathrm{d}[\xi_1, \xi_2]_{E^*} - \mathcal{L}_{\xi_1}\mathrm{d}\xi_2 + \mathcal{L}_{\xi_2}\mathrm{d}\xi_1). \end{aligned} \tag{A.43}$$

Subsequent calculations are simplified by focusing on the term $\iota_{X_3}\mathcal{L}_{\xi_1}\mathrm{d}\xi_2 - \iota_{X_3}\mathcal{L}_{\xi_2}\mathrm{d}\xi_1$ in the first stage. From the calculations in the previous section, it can be seen that, in general

$$\iota_X\mathcal{L}_\xi \mathrm{d}\eta = [\xi, \mathcal{L}_X\eta]_{E^*} - \mathcal{L}_{\mathcal{L}_\xi X}\eta + [\mathrm{d}\langle\eta, X\rangle, \xi]_{E^*} + \mathrm{d}(\rho_*(\xi)\langle\eta, X\rangle) - \mathrm{d}\langle[\xi, \eta]_{E^*}, X\rangle. \tag{A.18}$$

is satisfied. Substituting (A.18) into (A.43) , we obtain the following form.

$$\begin{aligned}\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*} = &-\mathrm{d}\langle[\xi_1,\xi_2]_{E^*},X_3\rangle\\&+[\xi_1,\mathcal{L}_{X_3}\xi_2]_{E^*}-\mathcal{L}_{\mathcal{L}_{\xi_1}X_3}\xi_2+[\mathrm{d}\langle\xi_2,X_3\rangle,\xi_1]_{E^*}+\mathrm{d}(\rho_*(\xi_1)\langle\xi_2,X_3\rangle)\\&-[\xi_2,\mathcal{L}_{X_3}\xi_1]_{E^*}+\mathcal{L}_{\mathcal{L}_{\xi_2}X_3}\xi_1-[\mathrm{d}\langle\xi_1,X_3\rangle,\xi_2]_{E^*}-\mathrm{d}(\rho_*(\xi_2)\langle\xi_1,X_3\rangle)\\&+\iota_{X_3}(\mathrm{d}[\xi_1,\xi_2]_{E^*}-\mathcal{L}_{\xi_1}\mathrm{d}\xi_2+\mathcal{L}_{\xi_2}\mathrm{d}\xi_1).\end{aligned}\tag{A.44}$$

Next, calculate the cyclic sum $\mathcal{L}_{X_3}[\xi_1,\xi_2]_{\mathrm{V}}+\text{c.p.}$ of the (A.44). The terms corresponding to the second and third stages of (A.44) can be summarised as follows.

$$[\xi_1,\mathcal{L}_{X_3}\xi_2]_{E^*}-[\xi_2,\mathcal{L}_{X_3}\xi_1]_{E^*}+\text{c.p.}=[\mathcal{L}_{X_1}\xi_2-\mathcal{L}_{X_2}\xi_1,\xi_3]_{E^*}+\text{c.p.},\tag{A.45}$$

$$-\mathcal{L}_{\mathcal{L}_{\xi_1}X_3}\xi_2+\mathcal{L}_{\mathcal{L}_{\xi_2}X_3}\xi_1+\text{c.p.}=\mathcal{L}_{\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1}\xi_3+\text{c.p.},\tag{A.46}$$

$$[\mathrm{d}\langle\xi_2,X_3\rangle,\xi_1]_{E^*}-[\mathrm{d}\langle\xi_1,X_3\rangle,\xi_2]_{E^*}+\text{c.p.}=+2[\mathrm{d}(e_1,e_2)_-,\xi_3]_{E^*}+\text{c.p.},\tag{A.47}$$

$$\mathrm{d}(\rho_*(\xi_1)\langle\xi_2,X_3\rangle)-\mathrm{d}(\rho_*(\xi_2)\langle\xi_1,X_3\rangle)+\text{c.p.}=2\mathrm{d}(\rho_*(\xi_3)\cdot(e_1,e_2)_-)+\text{c.p.}\tag{A.48}$$

therefore, we obtain

$$\begin{aligned}\mathcal{L}_{X_3}[\xi_1,\xi_2]_{E^*}+\text{c.p.}=&-\mathrm{d}\langle[\xi_1,\xi_2]_{E^*},X_3+[\mathcal{L}_{X_1}\xi_2-\mathcal{L}_{X_2}\xi_1,\xi_3]_{E^*}+\mathcal{L}_{\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1}\xi_3\\&+2[\mathrm{d}(e_1,e_2)_-,\xi_3]_{E^*}+2\mathrm{d}(\rho_*(\xi_3)\cdot(e_1,e_2)_-)\\&+\iota_{X_3}(\mathrm{d}[\xi_1,\xi_2]_{E^*}-\mathcal{L}_{\xi_1}\mathrm{d}\xi_2+\mathcal{L}_{\xi_2}\mathrm{d}\xi_1)+\text{c.p.}.\end{aligned}\tag{A.49}$$

This is the right-hand side of the(A.19).

A.4 Axiom C2

Since we cannot compute this directly, we act arbitrary $f\in C^\infty(M)$ for both side of (3.20). We check that the following equation holds.

$$\rho_{\mathrm{V}}([e_1,e_2]_{\mathrm{V}})f=[\rho_{\mathrm{V}}(e_1),\rho_{\mathrm{V}}(e_2)]f\tag{3.20'}$$

Expanding the left-hand side, from the definitions of ρ and $[\cdot,\cdot]_{\mathrm{V}}$, we obtain the following form.

$$\begin{aligned}\rho_{\mathrm{V}}([e_1,e_2]_{\mathrm{V}})f&=\rho_{\mathrm{V}}([X_1+\xi_1,X_2+\xi_2]_{\mathrm{V}})f\\&=\rho\{[X_1,X_2]_E+\mathcal{L}_{\xi_1}X_2-\mathcal{L}_{\xi_2}X_1-\mathrm{d}_*(e_1,e_2)_-\}f\\&\quad+\rho_*\{[\xi_1,\xi_2]_{E^*}+\mathcal{L}_{X_1}\xi_2-\mathcal{L}_{X_2}\xi_1+\mathrm{d}(e_1,e_2)_-\}f\\&=\rho([X_1,X_2]_E)f+\rho(\mathcal{L}_{\xi_1}X_2)f-\rho(\mathcal{L}_{\xi_2}X_1)f\\&\quad-\frac{1}{2}\rho\rho_*^*\mathrm{d}_0(\langle\xi_1,X_2\rangle-\langle\xi_2,X_1\rangle)f\\&\quad+\rho_*([\xi_1,\xi_2]_{E^*})f+\rho_*(\mathcal{L}_{X_1}\xi_2)f-\rho_*(\mathcal{L}_{X_2}\xi_1)f\\&\quad+\frac{1}{2}\rho_*\rho^*\mathrm{d}_0(\langle\xi_1,X_2\rangle-\langle\xi_2,X_1\rangle)f\end{aligned}\tag{A.50}$$

On the way, we used the fact that $\mathrm{d}_* = \rho_*^* \mathrm{d}_0, \mathrm{d} = \rho^* \mathrm{d}_0$. here, $[\rho(X_1), \rho(X_2)]$ と $[\rho_*(\xi_1), \rho_*(\xi_2)]$ becomes

$$\begin{aligned}
[\rho(X_1), \rho(X_2)] &= \rho([X_1, X_2]_E) \\
[\rho_*(\xi_1), \rho_*(\xi_2)] &= \rho_*([\xi_1, \xi_2]_{E^*})
\end{aligned}$$

from the difinition of ρ, ρ_*. So, we rewrite (A.50) as

$$\begin{aligned}
\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}})f = {} & [\rho(X_1), \rho(X_2)]f + \rho(\mathcal{L}_{\xi_1} X_2)f - \rho(\mathcal{L}_{\xi_2} X_1)f \\
& - \frac{1}{2}\rho\rho_*^* \mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f \\
& + [\rho_*(\xi_1), \rho_*(\xi_2)]f + \rho_*(\mathcal{L}_{X_1}\xi_2)f - \rho_*(\mathcal{L}_{X_2}\xi_1)f \\
& + \frac{1}{2}\rho_*\rho^* \mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f
\end{aligned} \tag{A.51}$$

Note that $[\cdot, \cdot]$ is a Lie bracket on TM. Also, we use

$$\begin{aligned}
[\rho(X), \rho_*(\xi)]f = {} & -\rho(\mathcal{L}_\xi X)f + \rho_*(\mathcal{L}_X \xi)f + (\rho\rho_*^* \mathrm{d}_0 \langle \xi, X \rangle)f \\
& - \langle \xi, \mathcal{L}_{\mathrm{d}f} X \rangle + \rho(X)\rho_*(\xi)f - \rho_*(\mathcal{L}_X \xi)f,
\end{aligned} \tag{A.52}$$

(A.51) is further organised,

$$\begin{aligned}
\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}})f = {} & [\rho(X_1), \rho(X_2)]f + \{\rho(\mathcal{L}_{\xi_1} X_2) - \rho_*(\mathcal{L}_{X_2}\xi_1)\}f \\
& - \frac{1}{2}\rho\rho_*^* \mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f \\
& + [\rho_*(\xi_1), \rho_*(\xi_2)]f - \{\rho(\mathcal{L}_{\xi_2} X_1) - \rho_*(\mathcal{L}_{X_1}\xi_2)\}f \\
& + \frac{1}{2}\rho_*\rho^* \mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f \\
= {} & [\rho(X_1), \rho(X_2)]f + \{\rho(\mathcal{L}_{\xi_1} X_2) - \rho_*(\mathcal{L}_{X_2}\xi_1) - \rho\rho_*^* \mathrm{d}_0 \langle \xi_1, X_2 \rangle\}f \\
& + \frac{1}{2}\rho\rho_*^* \mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f \\
& + [\rho_*(\xi_1), \rho_*(\xi_2)]f - \{\rho(\mathcal{L}_{\xi_2} X_1) - \rho_*(\mathcal{L}_{X_1}\xi_2) - \rho\rho_*^* \mathrm{d}_0 \langle \xi_2, X_1 \rangle\}f \\
& + \frac{1}{2}\rho_*\rho^* \mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f.
\end{aligned} \tag{A.53}$$

From this, $[\rho(X), \rho_*(\xi)]f$ can be summarised. The missing terms are added and subtracted to balance the books as follows.

$$\begin{aligned}
\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}})f = & [\rho(X_1), \rho(X_2)]f - [\rho(X_2), \rho_*(\xi_1)]f \\
& - \langle \xi_1, \mathcal{L}_{\mathrm{d}f} X_2 \rangle + \rho(X_2)\rho_*(\xi_1)f - \rho_*(\mathcal{L}_{X_2})\xi_1 f \\
& + \frac{1}{2}\rho\rho_*^*\mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f \\
& + [\rho_*(\xi_1), \rho_*(\xi_2)]f + [\rho(X_1), \rho_*(\xi_2)]f \\
& + \langle \xi_2, \mathcal{L}_{\mathrm{d}f} X_1 \rangle - \rho(X_1)\rho_*(\xi_2)f + \rho_*(\mathcal{L}_{X_1}\xi_2)f \\
& + \frac{1}{2}\rho_*\rho^*\mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f \\
= & [\rho(X_1), \rho(X_2)]f + [\rho_*(\xi_1), \rho(X_2)]f + [\rho_*(\xi_1), \rho_*(\xi_2)]f + [\rho(X_1), \rho_*(\xi_2)]f \\
& - \langle \xi_1, \mathcal{L}_{\mathrm{d}f} X_2 - [X_2, \mathrm{d}_* f]_E \rangle + \langle \xi_2, \mathcal{L}_{\mathrm{d}f} X_1 - [X_1, \mathrm{d}_* f]_E \rangle
\end{aligned} \tag{A.54}$$

Furthermore, from the definition of ρ, terms in the first line in (A.54) can be summarised as follows,

$$\begin{aligned}
\rho_{\mathrm{V}}([e_1, e_2]_{\mathrm{V}})f = & [\rho_{\mathrm{V}}(e_1), \rho_{\mathrm{V}}(e_2)] - \langle \xi_1, \mathcal{L}_{\mathrm{d}f} X_2 - [X_2, \mathrm{d}_* f]_E \rangle + \langle \xi_2, \mathcal{L}_{\mathrm{d}f} X_1 - [X_1, \mathrm{d}_* f]_E \rangle \\
& + \frac{1}{2}\rho\rho_*^*\mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f + \frac{1}{2}\rho_*\rho^*\mathrm{d}_0(\langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle)f.
\end{aligned} \tag{A.55}$$

In general, the right-hand side is not 0 for any X_i, ξ_1. Therefore, Axiom C2 is broken in $(\mathcal{V}, [\cdot, \cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot, \cdot)_+)$.

A.5 Axiom C3

We check the Axiom C3. Indeed, we expand the left-hand side $[e_1, fe_2]_{\mathrm{V}}$ and show that we arrive at the right-hand side. From the diffinition of the ρ,

$$\begin{aligned}
[e_1, fe_2]_{\mathrm{V}} &= [X_1 + f\xi_1, X_2, +f\xi_2]_{\mathrm{V}} \\
&= [X_1, fX_2]_{\mathrm{V}} + [X_1, f\xi_2]_{\mathrm{V}} + [\xi_1, fX_2]_{\mathrm{V}} + [\xi_1, f\xi_2]_{\mathrm{V}}.
\end{aligned} \tag{A.56}$$

here, from the difinition of the $[\cdot, \cdot]_{\mathrm{V}}$, we obtain

$$\begin{aligned}
[X_1, f\xi_2]_{\mathrm{V}} &= -\mathcal{L}_{f\xi_2} X_1 + \frac{1}{2}\mathrm{d}_*(f\langle \xi_2, X_1 \rangle) + \mathcal{L}_{X_1}(f\xi_2) - \frac{1}{2}\mathrm{d}(f\langle \xi_2, X_1 \rangle) \\
&= -f\mathcal{L}_{\xi_2} X_1 - (\mathrm{d}_* f)\langle \xi_2, X_1 \rangle + \frac{1}{2}(\mathrm{d}_* f)\langle \xi_2, X_1 \rangle + \frac{1}{2}f\mathrm{d}_*\langle \xi_2, X_1 \rangle \\
&\quad + f\mathrm{d}\langle \xi_2, X_1 \rangle + \iota_X \mathrm{d}f\xi_2 + f\iota_X\mathrm{d}\xi_2 - \frac{1}{2}(\mathrm{d}f)\langle \xi_2, X_1 \rangle - \frac{1}{2}f\mathrm{d}\langle \xi_2, X_1 \rangle \\
&= f[X_1, \xi_2]_{\mathrm{V}} + (\rho(X_1)f)\xi_2 - \frac{1}{2}\mathcal{D}f\langle \xi_2, X_1 \rangle.
\end{aligned} \tag{A.57}$$

Similarly for $[\xi_1, fX_2]_{\mathrm{V}}$, from the difinition of the $[\cdot, \cdot]_{\mathrm{V}}$, we obtain

$$[\xi_1, fX_2]_{\mathrm{V}} = f[\xi_1, X_2]_{\mathrm{V}} + (\rho_*(\xi_1)f)X_2 - \frac{1}{2}\mathcal{D}f\langle \xi_1, X_2 \rangle. \tag{A.58}$$

Also, since L and L^* are Lie algebroids respectively, it follows that

$$[X_1, fX_2]_{\mathrm{V}} = [X_1, fX_2]_E = f[X_1, X_2]_{\mathrm{V}} + (\rho(X_1)f)X_2, \tag{A.59}$$
$$[\xi_1, f\xi_2]_{\mathrm{V}} = [\xi_1, f\xi_2]_{E^*} = f[\xi_1, \xi_2]_V + (\rho_*(\xi_1)f)\xi_2. \tag{A.60}$$

therefore,

$$\begin{aligned}
[e_1, fe_2]_{\mathrm{V}} &= \text{(A.56)} \\
&= \text{(A.59)} + \text{(A.57)} + \text{(A.58)} + \text{(A.60)} \\
&= f[X_1, X_2]_{\mathrm{V}} + (\rho(X_1)f)X_2 \\
&\quad + f[X_1, \xi_2]_{\mathrm{V}} + (\rho(X_1)f)\xi_2 - \frac{1}{2}\mathcal{D}g\langle \xi_2, X_1\rangle \\
&\quad + f[\xi_1, X_2]_{\mathrm{V}} + (\rho_*(\xi_1)f)X_2 - \frac{1}{2}\mathcal{D}f\langle \xi_1, X_2\rangle \\
&\quad + f[\xi_1, \xi_2]_{\mathrm{V}} + (\rho_*(\xi_1)f)\xi_2 \\
&= f[e_1, e_2]_{\mathrm{V}} + (\rho_{\mathrm{V}}(e_1)f)e_2 - \mathcal{D}f(e_1, e_2)_-.
\end{aligned} \tag{A.61}$$

so, Axiom C3 is hold in $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$.

A.6 Axiom C4

Using the definition of $\mathcal{D}$, the left-hand side of (3.22) can be transformed as follows. Here, the external differential operator on $\Gamma(T^*M)$ is d_0.

$$\begin{aligned}
(\mathcal{D}f, \mathcal{D}g)_+ &= (\mathrm{d}f + \mathrm{d}_*f, \mathrm{d}g + \mathrm{d}_*g)_+ \\
&= \frac{1}{2}(\langle \mathrm{d}f, \mathrm{d}_*g\rangle + \langle \mathrm{d}g, \mathrm{d}_*f\rangle) \\
&= \frac{1}{2}(\rho_*(\mathrm{d}f)g + \rho(\mathrm{d}_*f)g) \\
&= \frac{1}{2}(\rho_*\rho^*\mathrm{d}_0 f + \rho\rho_*^*\mathrm{d}_0 f)g.
\end{aligned} \tag{A.62}$$

Therefore, $\rho_*\rho^* = -\rho\rho_*^*$ must hold for (A.62) to be 0 and for (3.22) to hold.

First, it is shown that if the derivation condition is imposed, the anchor ρ is always antisymmetric $\rho\rho_*^* = -\rho_*\rho^*$. However, as mentioned in the text, the superscript $*$ attached to the anchor means adjoint operator and is defined by the transpose of the original operator through the inner product.

$$\begin{aligned}
&\rho : E \to TM \qquad \rho^* : T^*M \to E^* \\
&\rho_* : E^* \to TM \quad \rho_*^* : T^*M \to E
\end{aligned} \tag{A.63}$$

Thus, $\rho\rho_*^* : T^*M \to TM$ and $\rho_*\rho^* : T^*M \to TM$.

In general, when an operator $\mathcal{O} : T^*M \to TM$ is antisymmetric, the following equation holds for all abitary $x \in \Gamma(T^*M)$.

$$\langle \mathcal{O}x, x\rangle = 0. \tag{A.64}$$

By replacing the operator $\mathcal{O}$ with $\rho\rho_*^*$and x with $\mathrm{d}_0 f \in \Gamma(TM)$, the following relation is obtained. Note that $f \in \Gamma(TM)$ and d_0 are external differential operators on T^*M.

$$\langle \rho\rho_*^*(\mathrm{d}_0 f), \mathrm{d}_0 f\rangle = \rho\rho_*^*(\mathrm{d}_0 f) \cdot f = 0. \tag{A.65}$$

Therefore, if (A.65) holds, then $\rho_*\rho^* = -\rho\rho_*^*$ holds.

Suppose that the following equation holds.

$$\mathcal{L}_{\mathrm{d}f} X + [\mathrm{d}_* f, X]_E = 0 \tag{3.53}$$

This equation is a concomitant expression from Proposition 3.4 of the reference [108], if the derivation condition is satisfied. Replacing X by $\mathrm{d}^* f$, the following relation holds..

$$\mathrm{d}_*\Big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\Big) = 0. \tag{A.66}$$

If f is replaced by f^2, then

$$\mathrm{d}_*\Big(\rho\rho_*^*(\mathrm{d}_0 f^2) \cdot f^2\Big) = 0. \tag{A.67}$$

On the other hand,

$$\rho\rho_*^*(\mathrm{d}_0 f^2) \cdot f^2 = \langle \mathrm{d}_0 f^2, \rho\rho_*^* \mathrm{d}_0 f^2\rangle = \langle \mathrm{d}f^2, \mathrm{d}_* f^2\rangle = 4f^2\langle \mathrm{d}f, \mathrm{d}_* f\rangle, \tag{A.68}$$

therefore,

$$\begin{aligned}\Big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\Big)\mathrm{d}_* f^2 &= \mathrm{d}_*\Big\{\big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\big)f^2\Big\} - \mathrm{d}_*\Big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\Big)f^2 \\ &= \frac{1}{4}\mathrm{d}_*\Big(\rho\rho_*^*(\mathrm{d}_0 f^2) \cdot f^2\Big) - \mathrm{d}_*\Big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\Big)f^2. \end{aligned} \tag{A.69}$$

here, some therms in right hand side is vanished by (A.66), (A.67),

$$\Big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\Big)\rho_*^* \mathrm{d}_0 f^2 = 0. \tag{A.70}$$

Furthermore, given that this is an inner product with $\mathrm{d}_0 f$,

$$2f\Big(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f\Big)^2 = 0. \tag{A.71}$$

Therefore, if derivation conditions are imposed, $0 = \rho\rho_*^*(\mathrm{d}_0 f) \cdot f = \langle \rho\rho_*^*(\mathrm{d}_0 f)$. For $\rho\rho_*^*$to be antisymmetric, a derivation condition must be imposed.

The following calculations are supplementary to the above results. In the general case where the derivation condition is not imposed, it is shown that the case where $\rho_*\rho^* = -\rho\rho_*^*$ does not hold cannot be excluded. Specifically, it is confirmed that $(\mathcal{V}, [\cdot,\cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot,\cdot)_+)$does not satisfy the relation (3.22). We deerive the left-hand side of (A.65) in a different way and show that it does not 0 in general.

From the properties of $[\cdot,\cdot]_E$, let $X, Y \in \Gamma(E)$,

$$\begin{aligned}\mathrm{d}_*[X, fY]_E &= \mathrm{d}_*(f[X,Y]_E + (\rho(X)\cdot f)Y) \\ &= \mathrm{d}_*(f[X,Y]_E) + \mathrm{d}_*((\rho(X)\cdot f)Y) \\ &= \mathrm{d}_* f \wedge [X,Y]_E + f[X,Y]_E + \mathrm{d}_*\rho(X)\cdot f \wedge Y + \rho(X)\cdot f \mathrm{d}_* Y \end{aligned} \tag{A.72}$$

is hold.Therefore, the following relationship holds.

$$\begin{aligned}[X, \mathrm{d}_* f]_E \wedge Y = \mathcal{L}_{\mathrm{d}f} X \wedge Y - \mathrm{d}_*[X, fY]_E + f\mathrm{d}_*[X, Y]_E \\ + \mathrm{d}_* f \wedge \mathcal{L}_X Y + \mathcal{L}_X \mathrm{d}_* f \wedge Y + (\rho(X) \cdot f)\mathrm{d}_* Y + f\mathcal{L}_Y \mathrm{d}_* X - f\mathcal{L}_{fY}\mathrm{d}_* X\end{aligned} \tag{A.73}$$

Replacing X by $\mathrm{d}_* f$ and rearranging, the following relation is obtained.

$$\begin{aligned}[\mathrm{d}_* f, \mathrm{d}_* f]_E \wedge Y &= \mathcal{L}_{\mathrm{d}f} \mathrm{d}_* f \wedge Y - \mathrm{d}_*[\mathrm{d}_* f, fY]_E + f\mathrm{d}_*[\mathrm{d}_* f, Y]_E \\ &\quad + \mathrm{d}_* f \wedge \mathcal{L}_{\mathrm{d}_* f} Y + (\rho(\mathrm{d}_* f) \cdot f)\mathrm{d}_* Y \\ &= 0\end{aligned} \tag{A.74}$$

Therefore, the following equation holds.

$$\mathcal{L}_{\mathrm{d}f} \mathrm{d}_* f \wedge Y = \mathrm{d}_*[\mathrm{d}_* f, fY]_E - f\mathrm{d}_*[\mathrm{d}_* f, Y]_E - \mathrm{d}_* f \wedge \mathcal{L}_{\mathrm{d}_* f} Y - (\rho(\mathrm{d}_* f) \cdot f)\mathrm{d}_* Y \tag{A.75}$$

Also, (A.75) hold if f is replaced by f^2.

$$\mathcal{L}_{\mathrm{d}f^2} \mathrm{d}_* f^2 \wedge Y = \mathrm{d}_*[\mathrm{d}_* f^2, f^2 Y]_E - f^2\mathrm{d}_*[\mathrm{d}_* f^2, Y]_E - \mathrm{d}_* f^2 \wedge \mathcal{L}_{\mathrm{d}_* f^2} Y - (\rho(\mathrm{d}_* f^2) \cdot f^2)\mathrm{d}_* Y. \tag{A.75$'$}$$

Furthermore, expanding from (A.65), we find that

$$\begin{aligned}\rho\rho_*^*(\mathrm{d}_0 f^2) \cdot f^2 &= \langle \rho\rho_*^*(\mathrm{d}_0 f^2), \mathrm{d}_0 f^2 \rangle \\ &= \langle \mathrm{d}_* f^2, \mathrm{d}f^2 \rangle \\ &= 4\langle f\mathrm{d}_* f, f\mathrm{d}f \rangle \\ &= 4f^2\langle \mathrm{d}_* f, \mathrm{d}f \rangle \\ &= 4(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f)f^2.\end{aligned} \tag{A.76}$$

So, if we act on both sides with d_*, we obtain

$$\begin{aligned}\mathrm{d}_*(\rho\rho_*^*(\mathrm{d}_0 f^2) \cdot f^2) &= 4\mathrm{d}_*((\rho\rho_*^*(\mathrm{d}_0 f) \cdot f)f^2) \\ &= 4f^2\mathrm{d}_*(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f) + 4(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f)\mathrm{d}_* f^2.\end{aligned} \tag{A.77}$$

The formula (A.73) can be rewritten as follows.

$$\begin{aligned}&4f^2\mathrm{d}_*(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f) \wedge Y + 4(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f)\mathrm{d}_* f^2 \wedge Y \\ &= \mathrm{d}_*[\mathrm{d}_* f^2, f^2 Y]_E + \big(-(\mathrm{d}_* f^2) \wedge \mathcal{L}_{\mathrm{d}_* f^2} Y - f\mathcal{L}_{\mathrm{d}_* f^2}\mathrm{d}_* Y - \rho(\mathrm{d}_* f^2)[f^2]\mathrm{d}_* Y\big).\end{aligned} \tag{A.75”}$$

Substituting (A.75) on the left-hand side, we obtain

$$\begin{aligned}&4f^2\mathrm{d}_*[\mathrm{d}_* f, fY]_E - 4f^2(\mathrm{d}_* f) \wedge \mathcal{L}_{\mathrm{d}_* f} Y - 4f^2\mathcal{L}_{\mathrm{d}_* f}\mathrm{d}_* Y - 4f^2(\rho(\mathrm{d}_* f) \cdot f)\mathrm{d}_* Y + 4(\rho\rho_*^*(\mathrm{d}_0 f) \cdot f)\mathrm{d}_* f^2 \wedge Y \\ &= \mathrm{d}_*[\mathrm{d}_* f^2, f^2 Y]_E - (\mathrm{d}_* f^2) \wedge \mathcal{L}_{\mathrm{d}_* f^2} Y - f\mathcal{L}_{\mathrm{d}_* f^2}\mathrm{d}_* Y - (\rho(\mathrm{d}_* f^2) \cdot f^2)\mathrm{d}_* Y.\end{aligned}$$

Summarising this in terms of certain terms in $\rho\rho_*^*(\mathrm{d}_0 f)\cdot f\mathrm{d}_* f^2$, we obtain

$$\begin{aligned}4\rho\rho_*^*(\mathrm{d}_0 f)\cdot f\mathrm{d}_* f^2\wedge Y &= -4f^2\mathrm{d}_*[\mathrm{d}_* f, fY]_E + 4f^2(\mathrm{d}_* f)\wedge\mathcal{L}_{\mathrm{d}_* f}Y \\ &\quad + 4f^2\mathcal{L}_{\mathrm{d}_* f}\mathrm{d}_* Y + 4f^2\rho(\mathrm{d}_* f)\cdot f\mathrm{d}_* Y \\ &\quad + \mathrm{d}_*[\mathrm{d}_* f^2, f^2 Y]_E - (\mathrm{d}_* f^2)\wedge\mathcal{L}_{\mathrm{d}_* f^2}Y \\ &\quad - f\mathcal{L}_{\mathrm{d}_* f^2}\mathrm{d}_* Y - (\rho(\mathrm{d}_* f^2)\cdot f^2)\mathrm{d}_* Y.\end{aligned} \tag{A.78}$$

We act ρ on both sides of (A.78), The left-hand side becomes

$$\begin{aligned}\rho((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)(\mathrm{d}_* f^2)) &= (\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)\rho(\mathrm{d}_* f^2) \\ &= (\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)\rho\rho_*^*(\mathrm{d}_0 f^2) \\ &= 2f((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)\rho\rho_*^*\mathrm{d}_0 f).\end{aligned} \tag{A.79}$$

Therefoer, we obtain

$$4\rho\rho_*^*(\mathrm{d}_0 f)\cdot f\mathrm{d}_* f^2\wedge Y = 8f((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)\rho\rho_*^*\mathrm{d}_0 f)\wedge Y. \tag{A.80}$$

(If $(\rho(X\wedge Y) = \rho(X)\wedge\rho(Y),)$ (A.78) can be written as follows.

$$\begin{aligned}8f(\rho\rho_*^*(\mathrm{d}_0 f)\cdot f\rho\rho_*^*\mathrm{d}_0 f)\wedge\rho(Y) &= -4f^2\rho(\mathrm{d}_*[\mathrm{d}_* f, fY]_E) + 4f^2\rho(\mathrm{d}_* f)\wedge\rho(\mathcal{L}_{\mathrm{d}_* f}Y) \\ &\quad + 4f^2\rho(\mathcal{L}_{\mathrm{d}_* f}\mathrm{d}_* Y) + 4f^2\rho(\mathrm{d}_* f)\cdot f\rho(\mathrm{d}_* Y) \\ &\quad + \rho(\mathrm{d}_*[\mathrm{d}_* f^2, f^2 Y]_E) - \rho((\mathrm{d}_* f^2))\wedge\rho(\mathcal{L}_{\mathrm{d}_* f^2}Y) \\ &\quad - f\rho(\mathcal{L}_{\mathrm{d}_* f^2}\mathrm{d}_* Y) - (\rho(\mathrm{d}_* f^2)\cdot f^2)\rho(\mathrm{d}_* Y).\end{aligned} \tag{A.81}$$

Furthermore, taking the inner product with $\mathrm{d}_0 f$ on both sides, we obtain

$$\begin{aligned}&8f\iota_{\mathrm{d}_0 f}(((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f),\rho\rho_*^*\mathrm{d}_0 f)\wedge\rho(Y)) \\ &= 8f\iota_{\mathrm{d}_0 f}((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f),\rho\rho_*^*\mathrm{d}_0 f)\wedge\rho(Y) - 8f((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f),\rho\rho_*^*\mathrm{d}_0 f)\wedge\iota_{\mathrm{d}_0 f}\rho(Y) \\ &= 8f(\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)^2\wedge Y - 8f((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f),\rho\rho_*^*\mathrm{d}_0 f)\wedge\iota_{\mathrm{d}_0 f}\rho(Y).\end{aligned} \tag{A.82}$$

therefore,

$$\begin{aligned}&8f(\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)^2\wedge Y \\ &= 8f((\rho\rho_*^*(\mathrm{d}_0 f)\cdot f)\rho\rho_*^*\mathrm{d}_0 f)\wedge i_{\mathrm{d}_0 f}\rho(Y) \\ &\quad - 4f^2\rho(\mathrm{d}_*[\mathrm{d}_* f, fY]_E) + 4f^2\rho(\mathrm{d}_* f)\wedge\rho(\mathcal{L}_{\mathrm{d}_* f}Y) \\ &\quad + 4f^2\rho(\mathcal{L}_{\mathrm{d}_* f}\mathrm{d}_* Y) + 4f^2\rho(\mathrm{d}_* f)\cdot f\rho(\mathrm{d}_* Y) \\ &\quad + \rho(\mathrm{d}_*[\mathrm{d}_* f^2, f^2 Y]_E) - \rho((\mathrm{d}_* f^2))\wedge\rho(\mathcal{L}_{\mathrm{d}_* f^2}Y) \\ &\quad - f\rho(\mathcal{L}_{\mathrm{d}_* f^2}\mathrm{d}_* Y) - (\rho(\mathrm{d}_* f^2)\cdot f^2)\rho(\mathrm{d}_* Y).\end{aligned} \tag{A.83}$$

Although there is a discrepancy of about a factor, the middle term of the (A.65) equation appears on the left-hand side. In general, the right-hand side is clearly not 0 for arbitrary f, Y, so the possibility that $\rho\rho_*^*(\mathrm{d}_0 f)\cdot f$ is not 0 cannot be excluded.

A.7 Axiom C5

We check the Axion C5

$$\rho(e)(e_1, e_2) = ([e, e_1]_{\mathrm{V}} + \mathcal{D}(e, e_1), e_2) + (e_1, [e, e_2]_{\mathrm{V}} + \mathcal{D}(e, e_2)). \tag{3.23}$$

This can be shown immediately by using (A.5). From (A.5),

$$\begin{aligned}
([e, e_1]_{\mathrm{V}}, e_2)_+ &= T(e, e_1, e_2) + \frac{1}{2}\rho_{\mathrm{V}}(e)(e_1, e_2)_+ - \frac{1}{2}\rho_{\mathrm{V}}(e_1)(e, e_2)_+ \\
(e_1, [e, e_2]_{\mathrm{V}})_+ &= T(e, e_2, e_1) + \frac{1}{2}\rho_{\mathrm{V}}(e)(e_2, e_1)_+ - \frac{1}{2}\rho_{\mathrm{V}}(e_2)(e, e_1)_+
\end{aligned}$$

Adding these two equations, we obtain the following relationship ($T(e, e_2, e_1)$ is fully anti-symmetric).

$$\begin{aligned}
&([e, e_1]_{\mathrm{V}}, e_2)_+ + (e_1, [e, e_2]_{\mathrm{V}})_+ \\
&= T(e, e_1, e_2) + T(e, e_2, e_1) + \rho_{\mathrm{V}}(e)(e_1, e_2)_+ - \frac{1}{2}\rho(e_1)(e, e_2)_+ - \frac{1}{2}\rho(e_2)(e, e_1)_+ \\
&= \rho_{\mathrm{V}}(e)(e_1, e_2)_+ - \frac{1}{2}\rho(e_1)(e, e_2)_+ - \frac{1}{2}\rho(e_2)(e, e_1)_+
\end{aligned} \tag{A.84}$$

We solve (A.84) for $\rho(e)(e_1, e_2)_+$ and obtain the following fomula.

$$\rho(e)(e_1, e_2)_+ = ([e, e_1]_{\mathrm{V}}, e_2)_+ + (e_1, [e, e_2]_{\mathrm{V}})_+ + \frac{1}{2}\rho_{\mathrm{V}}(e_1)(e, e_2)_+ + \frac{1}{2}\rho_{\mathrm{V}}(e_2)(e, e_1)_+ \tag{A.85}$$

Here, the definition of the ρ and $\mathcal{D}$, and using $e_i = X_i + \xi_i$,

$$\begin{aligned}
\frac{1}{2}\rho_{\mathrm{V}}(e_1)(e, e_2)_+ &= \frac{1}{2}\rho(X_1)(e, e_2)_+ + \frac{1}{2}\rho_*(\xi_1)(e, e_2)_+ \\
&= \frac{1}{2}(\langle X_1, \mathrm{d}(e, e_2)_+\rangle + \langle \xi_1, \mathrm{d}_*(e, e_2)_+\rangle) \\
&= (\mathrm{d}(e, e_2)_+ + \mathrm{d}_*(e, e_2)_+, X_1 + \xi_1)_+ \\
&= (\mathcal{D}(e, e_2)_+, e_1)_+.
\end{aligned} \tag{A.86}$$

Similary,

$$\frac{1}{2}\rho_{\mathrm{V}}(e_2)(e, e_1)_+ = (\mathcal{D}(e, e_1)_+, e_2)_+. \tag{A.87}$$

We substitute these into (A.85),

$$\begin{aligned}
\rho_{\mathrm{V}}(e)(e_1, e_2)_+ &= ([e, e_1]_{\mathrm{V}}, e_2)_+ + (e_1, [e, e_2]_{\mathrm{V}})_+ + (\mathcal{D}(e, e_2)_+, e_1)_+ + (\mathcal{D}(e, e_1)_+, e_2)_+ \\
&= ([e, e_1]_{\mathrm{V}} + \mathcal{D}(e, e_1), e_2) + (e_1, [e, e_2]_{\mathrm{V}} + \mathcal{D}(e, e_2))
\end{aligned} \tag{A.88}$$

(A.88) is exactly (3.23) itself. Therefore, Axiom C5 holds in $(\mathcal{V}, [\cdot, \cdot]_{\mathrm{V}}, \rho_{\mathrm{V}}, (\cdot, \cdot)_+)$.

A.8 $N_K = N_P + N_{\tilde{P}}$

The N_K is a quantity used to evaluate the integrability of the approximate para-complex structure, given by

$$N_P(X, Y) = \tilde{P}[P(X), P(Y)], \quad N_{\tilde{P}}(X, Y) = P[\tilde{P}(X), \tilde{P}(Y)] \tag{A.89}$$

In this case, we verify that $N_K(X,Y) = N_P(X,Y) + N_{\tilde{P}}(X,Y)$.

From the definition of $P, \tilde{P}$, $N_P(X,Y), N_{\tilde{P}}(X,Y))$ can be written down so that K is revealed as follows.

$$\begin{aligned}
N_P(X,Y) &= \tilde{P}[P(X), P(Y)] \\
&= \frac{1}{2}(1-K)\left[\frac{1}{2}(1+K)X, \frac{1}{2}(1+K)Y\right] \\
&= \frac{1}{2}(1-K)\frac{1}{4}[X+K(X), Y+K(Y)] \\
&= \frac{1}{2}(1-K)\frac{1}{4}\big(XY + XK(Y) + K(X)Y + K(X)K(Y) \\
&\qquad\qquad - YX - YK(X) - K(Y)X - K(Y)K(X)\big) \\
&= \frac{1}{2}(1-K)\frac{1}{4}([X,Y] + [X,K(Y)] + [K(X),Y] + [K(X),K(Y)]) \\
&= \frac{1}{8}([X,Y] + [X,K(Y)] + [K(X),Y] + [K(X),K(Y)]) \\
&\quad + \frac{1}{8}(-K[X,Y] - K[X,K(Y)] - K[K(X),Y] - K[K(X),K(Y)]).
\end{aligned}$$

Similarly,

$$\begin{aligned}
N_{\tilde{P}}(X,Y) &= P[\tilde{P}(X), \tilde{P}(Y)] \\
&= \frac{1}{2}(1+K)\left[\frac{1}{2}(1-K)X, \frac{1}{2}(1-K)Y\right] \\
&= \frac{1}{2}(1+K)\frac{1}{4}[X-K(X), Y-K(Y)] \\
&= \frac{1}{2}(1+K)\frac{1}{4}\big(XY - XK(Y) - K(X)Y + K(X)K(Y) \\
&\qquad\qquad - YX + YK(X) + K(Y)X - K(Y)K(X)\big) \\
&= \frac{1}{2}(1+K)\frac{1}{4}([X,Y] - [X,K(Y)] - [K(X),Y] + [K(X),K(Y)]) \\
&= \frac{1}{8}([X,Y] - [X,K(Y)] - [K(X),Y] + [K(X),K(Y)]) \\
&\quad + \frac{1}{8}(K[X,Y] - K[X,K(Y)] - K[K(X),Y] + K[K(X),K(Y)]).
\end{aligned}$$

Therefore,

$$\begin{aligned}
N_P(X,Y) + N_{\tilde{P}}(X,Y) &= \frac{1}{8}([X,Y] + [X,K(Y)] + [K(X),Y] + [K(X),K(Y)]) \\
&\quad + \frac{1}{8}(-K[X,Y] - K[X,K(Y)] - K[K(X),Y] - K[K(X),K(Y)]) \\
&\quad + \frac{1}{8}([X,Y] - [X,K(Y)] - [K(X),Y] + [K(X),K(Y)]) \\
&\quad + \frac{1}{8}(K[X,Y] - K[X,K(Y)] - K[K(X),Y] + K[K(X),K(Y)]) \\
&= \frac{1}{4}([X,Y] - K[X,K(Y)] - K[K(X),Y] + [K(X),K(Y)]) \\
&= N_K(X,Y). \qquad \text{(A.90)}
\end{aligned}$$

hence, if $N_P(X,Y), N_{\tilde{P}}(X,Y)$ as

$$N_P(X,Y) = \tilde{P}[P(X), P(Y)], \quad N_{\tilde{P}}(X,Y) = P[\tilde{P}(X), \tilde{P}(Y)], \tag{A.91}$$

$N_K = N_P + N_{\tilde{P}}$ is satisfied.

List of Publications

- Haruka Mori, Shin Sasaki, and Kenta Shiozawa,“Doubled Aspects of Vaisman Algebroid and Gauge Symmetry in Double Field Theory,” J. Math. Phys. 61 (2020) 013505, [arXiv:1901.04777].

- Haruka Mori, Shin Sasaki, and Kenta Shiozawa, “Vaisman Algebroid and Doubled Structure of Gauge Symmetry in Double Field Theory,” J. Phys. Conf. Ser. 1416 (2019) 012023.

- Haruka Mori and Shin Sasaki, “More on Doubled Aspects of Algebroids in Double Field Theory,” J. Math. Phys. 61 (2020) 123504, [arXiv:2008.00402].

- Haruka Mori, Shin Sasaki and Kenta Shiozawa, “Doubled aspects of algebroids and gauge symmetry in double field theory”, Springer Proc. Math. Stat. (in press).

- M. Hatsuda, H. Mori, S. Sasaki and M. Yata, “Gauged Double Field Theory, Current Algebras and Heterotic Sigma Models,” [arXiv:2212.06476].

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